

Veiled singularities for the spherically symmetric massless Einstein-Vlasov system.

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Abstract

This paper continues the investigation of the formation of naked singularities in the collapse of collisionless matter initiated in [17]. There the existence of certain classes of non-smooth solutions of the Einstein-Vlasov system was proved. Those solutions are self-similar and hence not asymptotically flat. To obtain solutions which are more physically relevant it makes sense to attempt to cut off these solutions in a suitable way so as to make them asymptotically flat. This task, which turns out to be technically challenging, will be carried out in this paper.

1 Introduction

In this paper we continue the construction of a class of singular solutions of the Einstein-Vlasov system which was started in [17]. It is well known that solutions of the Einstein equations coupled with suitable matter models can yield singularities in finite time. More precisely, the meaning of this statement is the following. Solutions of the Einstein equations are certain spacetime manifolds, which are characterized by suitable pseudoriemannian metrics. The usual terminology in general relativity is that it is said that there is a singularity if the corresponding spacetime has a metric that fails to be causally geodesically complete. By this it is understood that there is a timelike or null geodesic which, at least in one direction, cannot be extended and has a finite affine length.

One of the best known examples of singularities in general relativity is given by black holes. They are characterized by the presence of a event horizon which ensures that the singularity does not have an influence on distant observers. A singularity which is not covered by an event horizon, so that it is visible to distant observers, is known as a naked singularity. This type of singularity is physically problematic for the following reason. To say that the singularity is visible to distant observers means that there exist causal geodesics which approach the singularity in the past time direction and which in the future time direction enter regions where the density of matter and the gravitational fields become arbitrarily small and the geometry resembles that of the flat Minkowski spacetime. These physical notions can be formulated mathematically using the

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concept of an asymptotically flat spacetime. In general relativity causal influences propagate along causal geodesics and so a naked singularity leads to a situation where the singularity can have an influence on physical processes in distant regions. Since we do not have a complete theory describing the physics at a singularity this means a breakdown of the ability of physics to make predictions. There is a situation which is a priori milder but which is problematic for a similar reason. This is where although there are no geodesics of the type characterizing a naked singularity there are families of causal geodesics which reach distant regions and come from regions where physical conditions are arbitrarily extreme. This can be made mathematically precise by saying that they come from regions where some geometrical invariants, such as the Kretschmann scalar $R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ or the invariant $T^{\alpha\beta}T_{\alpha\beta}$ built out of the energy-momentum tensor, become arbitrarily large. From the point of view of physics this is just as problematic as a naked singularity since we do not have a reliable theoretical control of physical phenomena in regimes far beyond those accessible to experiments. We denote this type of situation as a 'veiled singularity' since it means intuitively that although there is not a singularity which is directly visible we can nevertheless make observations of a region where known theory is in danger of being invalid. This terminology is not intended to indicate that a veiled singularity need be associated with geodesic incompleteness. A situation like this is also present in the solutions of the Einstein equations coupled to a massless scalar field with singular future light cones found by Christodoulou [6], [7]. Note, however, that Christodoulou's solutions have a significantly different causal structure from those we construct.

There are three features of the solutions constructed in [17] which could be seen as disadvantages from the point of view of their physical applicability and which should be improved if possible. The first is the fact which has already been mentioned that they are not asymptotically flat and thus do not represent isolated systems. The second is that the initial data from which they evolve is not smooth. The third is that the matter source used models massless particles like photons rather than ordinary matter. In this paper we remove the first disadvantage, leaving the other two for future investigation. The hope is that eventually the rough data can be approximated by smooth data and the massless particles by massive ones in such a way that the key dynamical properties of the solutions are preserved. In fact there is recent work which indicates that the case of massless particles may be of considerable interest in its own right as a physical model. There has been a detailed numerical study of gravitational collapse in that case which concentrates on type I critical collapse [1]. Type II critical collapse, which might be related to the phenomena studied in the present paper, is only briefly mentioned in [1]. The fact that global existence for small initial data has been proved for asymptotically flat solutions of the Einstein-Vlasov system with massless particles [18] is a strong indication of the robustness of this model.

Next the basic mathematical set-up used in the paper will be described. In general relativity the dynamics of self-gravitating matter is described by means of solutions of the Einstein equations coupled to other equations describing

the matter content. The type of singularities which could arise depend very strongly on the type of matter model used in the system. In this paper we will be concerned with collisionless matter, described by the Vlasov equation. Moreover, we will assume in addition that the point particles represented by the matter model have zero mass. The combined system of the Einstein equations with this model of matter is the massless Einstein-Vlasov system.

In what follows we will consider solutions of the Einstein-Vlasov system where the particle density is not a bounded function, but a measure concentrated on some hypersurfaces that will be described in detail later. A consequence of this is that the Einstein equations are not satisfied in classical form, but in a suitable distributional sense. There is a class of distributional solutions of the Einstein-Vlasov system which are equivalent to what is usually known in the literature as dust. Dust solutions of Vlasov systems have the property that there is a unique possible value of the velocity at each point of spacetime. The solutions considered in this paper are somehow more general than the usual dust solutions considered in the literature, because they have a set of admissible velocities at each point of the spacetime but the dimension of that admissible set of velocities is smaller than the total dimension of the phase space. Moreover, the set of admissible velocities at a given spatial point is qualitatively different in different regions. The number of possible values of the radial velocity for fixed angular momentum is two, one or zero. The number changes at some particular points which will be referred to as turning points. There the support of the distribution function fails to be transverse to the fibres of the tangent bundle. Some matter variables like the density and pressure become unbounded in a neighbourhood of the turning points. From this point of view the solutions considered in this paper are intermediate between dust and smooth solutions and hence will be called dust-like solutions. Note that in contrast to dust they do have some velocity dispersion. The dimension of the support of f in the tangent space at a given spacetime point is zero for dust, two for the solutions in this paper and three for smooth solutions of the Einstein-Vlasov system. There is a particular type of solutions of the Einstein equations known as generalized Einstein clusters, which were first studied in [2], [8]. These solutions can also be thought of as distributional solutions of the Einstein-Vlasov system for which the support of f in the tangent space at a given spacetime point is one. A more detailed discussion about the relation between generalized Einstein clusters and the Einstein-Vlasov system can be found in [17]. For the solutions here it will be possible to describe the distribution of velocities for the particles at a given point using a function depending on one coordinate, while a general distribution of velocities compatible with the assumption of spherical symmetry would depend on two coordinates.

The results of this paper are a continuation of those in [17]. In that paper, a class of dust-like self-similar solutions of the Einstein-Vlasov system were obtained. Those solutions do not have an event horizon anywhere in the spacetime. The difficulty with those solutions is that, due to their self-similar character, they cannot asymptotically resemble the Minkowski metric at large distances from the center. In this paper we show that it is possible to cut off the distri-

bution of matter in a suitable manner outside a compact set and to obtain a solution of the Einstein-Vlasov system whose metric behaves asymptotically far away from the center like that of Minkowski spacetime. In addition, the spacetime constructed in this paper will have the property that it is not geodesically complete and that horizons do not appear at any point of the spacetime.

It is interesting to remark that for the solutions constructed in this paper, the distribution of matter away from the center for long times asymptotically approaches the distribution of one Einstein cluster with all its mass contained in an interval $r \in (0, R_{\max})$, with a density of matter ρ which increases linearly with the distance to the center. A more detailed description of the matter distribution for long times, away from the center, will be found in Section 8.

Due to the mathematical complexity of the Einstein equations many of the studies related to singularity formation for these equations have been carried out for spherically symmetric solutions. In spherical symmetry the Einstein vacuum equations are non-dynamical due to Birkhoff's theorem, which says that any spherically symmetric vacuum solution is locally isometric to the Schwarzschild solution and, in particular, static. Thus it is essential to include matter of some kind. A matter model which has proved very useful for this task is the scalar field. This is a real-valued function ϕ which satisfies the wave equation $\nabla^\alpha \nabla_\alpha \phi = 0$. In this case the Einstein equations take the form $R_{\alpha\beta} = 8\pi \nabla_\alpha \phi \nabla_\beta \phi$ where $R_{\alpha\beta}$ is the Ricci curvature of $g_{\alpha\beta}$. The spherically symmetric Einstein-scalar field equations were studied in great detail in a series of papers by D. Christodoulou. This culminated in [6] and [7]. In [6] it was shown that in this system naked singularities can evolve from regular asymptotically flat initial data. It was shown in [7] that generic initial data do not lead to naked singularities. (A precise definition of naked singularities can be found in [17]).

For the spherically symmetric Einstein-scalar field equations it is known from the work of Christodoulou [5] that small asymptotically flat initial data lead to a solution which is geodesically complete and hence free of singularities. This small data result has recently been extended to the case without symmetry in [13]. On the other hand there are certain large initial data for which it is known that a black hole is formed. The threshold between these two types of behaviour was studied by Choptuik (cf. [4]) and many other papers since. This area of research is known as critical collapse. It is entirely numerical and heuristic and unfortunately mathematically rigorous results are not yet available.

The plan of this paper is the following. In Section 2 we recall the system of partial differential equations which describes the Einstein-Vlasov system in the spherically symmetric case. We also proved that it is possible to reformulate the problem as a system of equations where the angular momentum does not appear explicitly and therefore the system can be reformulated in terms of one variable less. Section 3 describes in a heuristic manner the construction which will be carried out in this paper and we state the main results. In this section we also give the precise definition of measured-valued solution which will be used in this paper. Section 4 summarizes the main properties of the self-similar solutions constructed in [17]. Moreover, some additional asymptotic

properties of the solution are also obtained. Section 5 contains a description of the functional analysis properties which will be used in the proof of the main results of the paper. This section also contains the description of some auxiliary PDEs which will be used in the proof of the main result. Section 6 contains the fixed point argument which proves the main result of the paper. Section 7 contains a description of the properties of the spacetime constructed in the paper. In particular, it is proved that the resulting metric is not geodesically complete and the absence of a horizon in the spacetime obtained.

In order to simplify the notation we will use the following convention. We will use generic functions Φ depending on the variables (t, r) , i.e. $\Phi = \Phi(t, r)$. However, on several occasions we will change to new variables (τ, y) by means of suitable diffeomorphisms $(t, r) \rightarrow (\tau, y)$. This defines a new function $\tilde{\Phi}$ satisfying $\tilde{\Phi}(\tau, y) = \Phi(t, r)$. We will denote $\tilde{\Phi}$ by Φ for simplicity, since no risk of confusion will arise due to this.

2 REFORMULATING THE EINSTEIN-VLASOV SYSTEM AS A SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS.

2.1 Einstein-Vlasov System in Schwarzschild coordinates.

We recall here the system of partial differential equations which describe the solutions of the massless Einstein-Vlasov system in the spherically symmetric case. These equations have been summarized in [15] and we will just refer to the corresponding formulas there.

A convenient way of writing the metric for spherically symmetric spacetimes uses a modified version of the classical Schwarzschild coordinates (cf. [15]):

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (2.1)$$

which is chosen so that $\mu(t, 0) = 0$. This normalization implies that the time variable t is just the proper time at the center $r = 0$.

Due to the symmetry of the metric, a suitable way to describe the kinematic characteristics of collisionless matter is by means of the following quantities:

$$r = |x|, \quad w^{(t)} = \frac{x \cdot v}{r}, \quad F = |x \wedge v|^2 \quad (2.2)$$

where the upper index (t) in $w^{(t)}$ stands for the tangential component.

A convenient feature of the choice of variables (2.2) is that the angular momentum variable F is constant along characteristics.

We will write the particle density as:

$$f = f(t, r, w^{(t)}, F)$$

Then, using the fact that collisionless matter moves along the light rays associated to the metric (2.1) we obtain, that the particle density satisfies the first order PDE (cf. [15]):

$$\partial_t f + e^{\mu-\lambda} \frac{w^{(t)}}{E} \partial_r f - \left(\lambda_t w^{(t)} + e^{\mu-\lambda} \mu_r E - e^{\mu-\lambda} \frac{F}{r^3 E} \right) \partial_{w^{(t)}} f = 0 \quad (2.3)$$

where:

$$E = \sqrt{(w^{(t)})^2 + \frac{F}{r^2}} \quad (2.4)$$

We are assuming in (2.4) that we are dealing with massless particles. For massive particles we should replace (2.4) by $E = \sqrt{1 + (w^{(t)})^2 + \frac{F}{r^2}}$, but in that case some invariance properties under rescalings which will be used in the construction of the solutions in this paper would be lost. Notice, however, that in the case of particles with velocities close to the speed of light (2.4) would be a good approximation for the particle energy, even in the case of massive particles.

Using the energy-momentum tensor for collisionless matter (cf. [15], [16]), it turns out that Einstein equations for gravitational fields become the following system of equations:

$$e^{-2\lambda} (2r\lambda_r - 1) + 1 = 8\pi r^2 \rho, \quad (2.5)$$

$$e^{-2\lambda} (2r\mu_r + 1) - 1 = 8\pi r^2 p \quad (2.6)$$

where, using suitable normalizations for t and r , we must use the following boundary conditions:

$$\mu(t, 0) = 0 \quad , \quad \lambda(t, 0) = 0 \quad , \quad (2.7)$$

$$\lambda(t, \infty) = 0. \quad (2.8)$$

The functions ρ , p in (2.5) encode all the relevant information in the energy-momentum tensor. In the case of collisionless matter they are given by:

$$\rho = \rho(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \left[\int_0^{\infty} E f dF \right] dw^{(t)}, \quad (2.9)$$

$$p = p(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \left[\int_0^{\infty} \frac{(w^{(t)})^2}{E} f dF \right] dw^{(t)}. \quad (2.10)$$

It is useful to notice that $p \leq \rho$. The system (2.3), (2.5)-(2.10), (2.4) is invariant under the rescaling:

$$r \rightarrow \theta r \quad , \quad t \rightarrow \theta t \quad \text{for } t < 0 \quad , \quad w^{(t)} \rightarrow \frac{1}{\sqrt{\theta}} w^{(t)} \quad , \quad F \rightarrow \theta F \quad (2.11)$$

for any $\theta > 0$. It is then natural to look for solutions of (2.3), (2.5)-(2.10) invariant under the rescaling (2.11). Such solutions are the self-similar solutions

studied in [17]. However, the metric associated to those solutions is not asymptotically flat as $r \rightarrow \infty$. The solutions obtained in this paper will be obtained by replacing the distribution of collisionless particles for $r \geq R_+(t)$, with R_+ of order one, by another distribution, with a smaller number of particles. As a consequence, it will not be possible to analyze the differential equations for the particle distribution f by means of ODEs, but on the contrary, a more involved analysis, which requires understanding the behaviour of some hyperbolic systems, will be required. This analysis will be the main contribution of this paper.

2.2 Elimination of the angular momentum for a massless system.

Due to the fact that we consider a system of massless particles, we can reformulate (2.3)-(2.10) as a system of equations where the variable F does not appear. More precisely, we can obtain a simpler, but equivalent PDE system, where the unknown function f depending on the variables $(t, w^{(t)}, r, F)$ is replaced by a new function ζ which depends only on three variables (t, v, r) . To this end we define a new variable:

$$v = \frac{w^{(t)}}{\sqrt{F}}. \quad (2.12)$$

Making the change of variables $(t, r, w^{(t)}, F) \rightarrow (t, r, v, F)$ and denoting the new distribution function by f with a slight abuse of notation we can transform the system (2.3)-(2.10) into:

$$\partial_t f + e^{\mu-\lambda} \frac{v}{\tilde{E}} \partial_r f - \left(\lambda_t v + e^{\mu-\lambda} \mu_r \tilde{E} - e^{\mu-\lambda} \frac{1}{r^3 \tilde{E}} \right) \partial_v f = 0, \quad (2.13)$$

$$\tilde{E} = \sqrt{v^2 + \frac{1}{r^2}}, \quad \rho = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \left[\int_0^{\infty} \tilde{E} f F dF \right] dv, \quad p = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \left[\int_0^{\infty} f \frac{v^2}{\tilde{E}} F dF \right] dv. \quad (2.14)$$

Notice that the change of variables (2.12) eliminates the dependence on the variable F for the characteristic curves associated to the Vlasov equation (cf. (2.13)). Moreover, the functions ρ and p and therefore the functions λ , μ characterizing the gravitational fields depend on f only through the reduced distribution function:

$$\zeta(t, r, v) \equiv \int_0^{\infty} f F dF. \quad (2.15)$$

In particular, it is possible to write a closed problem for the reduced distribution function that can be obtained by multiplying (2.13) by F and integrating

with respect to this variable:

$$\partial_t \zeta + e^{\mu-\lambda} \frac{v}{\tilde{E}} \partial_r \zeta - \left(\lambda_t v + e^{\mu-\lambda} \mu_r \tilde{E} - e^{\mu-\lambda} \frac{1}{r^3 \tilde{E}} \right) \partial_v \zeta = 0, \quad (2.16)$$

$$\tilde{E} = \sqrt{v^2 + \frac{1}{r^2}}, \quad \rho = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \tilde{E} \zeta dv, \quad p = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \frac{v^2}{\tilde{E}} \zeta dv. \quad (2.17)$$

This system must be complemented with the field equations (2.5), (2.6).

Notice that, given any solution of the problem (2.5), (2.6), (2.16), (2.17) we can obtain a solution of the original system (2.13), (2.14) by choosing any function $f = f(t, r, v, F)$ that gives the values of $\zeta(t_0, r, v)$ for any $t_0 \in \mathbb{R}$, by means of (2.15). Using the characteristics associated to (2.16), complemented with the equation $\frac{dF}{dt} = 0$ it is then possible to define $f(t, r, v, F)$ for the same range of values of (t, r, v) as the distribution ζ . These results are made completely precise in Propositions 17, 18 below.

3 GENERAL STRATEGY AND MAIN RESULTS.

3.1 Heuristic idea behind the construction.

The solution that we construct in this paper is obtained by means of a suitable perturbation of the self-similar solutions obtained in [17]. More precisely, our goal is to obtain measures $f = f(t, r, v, F)$, $\zeta = \zeta(t, r, v)$ solving (2.5), (2.6), (2.13), (2.14), (2.16), (2.17) in a suitable distributional sense (cf. Definitions 8, 13 below). The support of ζ consists of two surfaces in the space (r, v, t) having the form $\gamma_1(t) = \{v = v_1(t, r), r \geq y_0(-t), t_0 \leq t < 0\}$, $\gamma_2(t) = \{v = v_2(t, r), r \geq y_0(-t), t_0 \leq t < 0\}$ for some suitable functions $v_1 \leq v_2$ and numbers y_0, t_0 to be fixed. Therefore, the solutions constructed in this paper will have the form:

$$f(t, r, v, F) = A_1(t, r, F) \delta(v - v_1(t, r)) + A_2(t, r, F) \delta(v - v_2(t, r)) \quad (3.1)$$

$$\zeta(t, r, v) = B_1(t, r) \delta(v - v_1(t, r)) + B_2(t, r) \delta(v - v_2(t, r)) \quad (3.2)$$

where $A_1 \geq 0$, $A_2 \geq 0$, $B_1 > 0$, $B_2 \geq 0$.

We will assume in the rest of the paper that $f = 0$ for $(t, r, v, F) = (t, r, v, 0)$ in order to avoid singularities in (2.12). Actually, we will assume an even more stringent condition on f , namely $f = 0$ for $0 \leq F \leq \delta_0(-t)$ for some $\delta_0 > 0$. Concerning the support in the r coordinate, the self-similar solutions will vanish for $r \leq y_0(-t)$ for some $y_0 > 0$.

The self-similar solution constructed in [17] is a solution of (2.5), (2.6), (2.16), (2.17) with the form:

$$\zeta(t, r, v) = (-t)^2 \Theta(y, V), \quad \mu(t, r) = U(y), \quad \lambda(t, r) = \Lambda(y), \quad (3.3)$$

$$y = \frac{r}{(-t)}, \quad V = (-t)v \quad (3.4)$$

where the measure $\Theta(y, V)$ is supported in two curves in the plane $\{(y, V) : y > 0, V \in \mathbb{R}\}$ given by $\Gamma_1 = \{V = V_1(y) : y \geq y_0\}$, $\Gamma_2 = \{V = V_2(y) : y \geq y_0\}$, with $y_0 > 0$ and $V_1(y) < V_2(y)$ for $y > y_0$.

Due to its self-similar character, the solution obtained in [17] does not define a spacetime which is asymptotically flat as $r \rightarrow \infty$. We describe in this paper a procedure that allows to cut off this self-similar solution for sufficiently large radii r and obtain in this way an asymptotically flat spacetime. The rationale behind the cutoff procedure used is that in the limit $t \rightarrow 0^-$ most of the mass is concentrated in the curve Γ_2 . Actually, the fact that the spacetime associated to the self-similar solution is not asymptotically flat as $r \rightarrow \infty$ is due to the infinite amount of mass contained in Γ_2 . On the other hand the curve Γ_1 contains a small fraction of the mass that tends to zero as $t \rightarrow 0^-$. It is then natural, in order to obtain a solution containing a finite amount of mass, to cut off the branch Γ_2 at some value of the radius $r = R_+(t)$. Due to the fact that in spherically symmetric situations the gravitational fields at a given radius \bar{r} depend only in the distribution matter at radii $r \leq \bar{r}$ it follows that the dynamics of the branches Γ_1, Γ_2 is not modified for $r \leq R_+(t)$ and it agrees with the dynamics obtained for the self-similar solutions. However, the gravitational fields are modified for $r > R_+(t)$ and as a result the dynamics of the particles placed in the branch Γ_1 must be modified for $r > R_+(t)$. Our construction will then provide a measured-valued solution of (2.5), (2.6), (2.16), (2.17) supported in the union of two curves $\gamma_1(t), \gamma_2(t) \subset \{(r, v) : r > 0, v \in \mathbb{R}\}$ with $t_0 \leq t < 0$ satisfying:

$$\mathcal{U}(t)\gamma_1(t) = \Gamma_1 \quad \text{if } r \leq R_+(t), \quad \mathcal{U}(t)\gamma_2(t) = \Gamma_2 \cap \left\{y \leq \frac{R_+(t)}{(-t)}\right\} \quad (3.5)$$

$$\lim_{t \rightarrow 0^-} R_+(t) = R_{\max} > 0 \quad (3.6)$$

where $\mathcal{U}(t)$ is a transformation from the half-plane $\{(r, v) : r > 0, v \in \mathbb{R}\}$ to $\{(y, V) : y > 0, V \in \mathbb{R}\}$ given by (3.4) for any $t < 0$.

Notice that the intersection of the support of the solution obtained with the set $\{r > R_+(t)\}$ is just $\gamma_1(t) \cap \{r > R_+(t)\}$. The solution obtained is not self-similar for $r > R_+(t)$ due to the presence in the problem of a length scale R_{\max} . On the other hand it is also worth noticing that due to the change of the spacetime structure for $r > R_+(t)$ the most suitable time variable to describe the dynamics of the region $r > R_+(t)$ is not t , but the new time variable $\tau = -\log(-t)$ that basically corresponds to the Minkowski time for $r \rightarrow \infty$. In this time variable the formation of the singularity takes place for proper times approaching infinity. On the contrary, at the center $r = 0$, the formation of the singularity will take place in finite proper time.

Notation 1 From now on we will denote as C a positive constant independent of the variables $\tau, \bar{\tau}, t, r, L, T$. However, the constant C could depend on y_0, R_{\max} . Some of these variables will be defined later.

3.2 Definition of measure-valued solutions.

We need to make precise in which sense the measure $f = f(t, r, v, F)$ defines a solution of the Einstein-Vlasov system (2.3)-(2.10), (2.13), (2.14). The definition that we will give in this Section has the advantage that it only requires few regularity conditions for the functions λ, μ or equivalently for the densities ρ, p .

As a first step, we need the following auxiliary result concerning the well-posedness of the problem (2.5)-(2.8).

Lemma 2 *Suppose that $r^2\rho, r^2p \in L^1(\mathbb{R}_+)$. Let us assume also that the function $R_0(r) = 8\pi \int_0^r \xi^2 \rho(\xi) d\xi$ satisfies:*

$$R_0(r) < r \text{ if } r \in (0, \infty), \lim_{r \rightarrow 0} \frac{R_0(r)}{r} = 0, \int_0^1 \frac{R_0(\xi)}{\xi^2} d\xi < \infty, \int_0^1 \xi p(\xi) d\xi < \infty \quad (3.7)$$

We define functions λ and μ by means of:

$$\lambda(r) = \frac{1}{2} \log(r) - \frac{1}{2} \log(r - R_0(r)) \quad (3.8)$$

$$\mu(r) = \int_0^r \frac{4\pi \xi^2 p(\xi) d\xi}{(\xi - R_0(\xi))} + \frac{1}{2} \int_0^r \frac{R_0(\xi) d\xi}{(\xi - R_0(\xi)) \xi} \quad (3.9)$$

Then the functions λ, μ are in $C[0, \infty) \cap W_{\text{loc}}^{1,1}(0, \infty)$ and satisfy (2.7), (2.8). They solve (2.5), (2.6) for almost all $r \in (0, \infty)$.

For any $\delta_0 > 0$, let us denote as \mathcal{Z}_{δ_0} the set of functions in $(L^1(\mathbb{R}_+, r^2 dr))^2$ which are supported in the half line $[\delta_0, \infty)$ and satisfy $R_0(r) < r$ for any $r \in (0, \infty)$. Let us endow \mathcal{Z}_{δ_0} with the topology of $(L^1(\mathbb{R}_+, r^2 dr))^2$. Then, the mapping $(\rho, p) \rightarrow (\lambda, \mu)$ defines a continuous mapping from \mathcal{Z}_{δ_0} to $(W^{1,1}(0, L))^2$ for any $L > 0$.

Proof. The conditions (3.7) imply that the functions λ and μ in (3.8), (3.9) are well defined for $r > 0$ and they belong to $W_{\text{loc}}^{1,1}(0, \infty)$. Then, they are continuous in $(0, \infty)$ and they also satisfy:

$$\lim_{r \rightarrow 0^+} \lambda(r) = 0, \quad \lim_{r \rightarrow 0^+} \mu(r) = 0 \quad (3.10)$$

whence $\lambda, \mu \in C[0, \infty)$. We can differentiate λ and μ for almost all $r \in (0, \infty)$ and check by means of one explicit computation that they solve (2.5), (2.6) a.e. $r \in (0, \infty)$.

It only remains to check the continuity of the mapping $(\rho, p) \rightarrow (\lambda, \mu)$ defined from \mathcal{Z}_{δ_0} to $(W^{1,1}(0, \infty))^2$. Suppose that $(\bar{\rho}, \bar{p}) \in \mathcal{Z}_{\delta_0}$ and let us write $\bar{R}_0(r) = 8\pi \int_0^r \xi^2 \bar{\rho}(\xi) d\xi$. By assumption $\bar{R}_0(r) = 0$ if $r \leq \delta_0$ and $\bar{R}_0(r) < r$ if $r > \delta_0$. Moreover, since $\bar{\rho} \in L^1(\mathbb{R}_+, r^2 dr)$ we have that $\bar{R}_0(r)$ is bounded for large r . Then, there exists $\eta > 0$ small such that $\bar{R}_0(r) < (1 - 2\eta)r$ if $r > \delta_0$. If we choose ρ, p supported in $\{r \geq \delta_0\}$ such that $\int_0^\infty |\rho(r) - \bar{\rho}(r)| r^2 dr$ is small, it then follows that $R_0(r) = 0$ if $r \leq \delta_0$ and $R_0(r) < (1 - \eta)r$ if $r > \delta_0$. Moreover,

we have also $\sup_{r \geq \delta_0} \frac{|R_0(r) - \bar{R}_0(r)|}{r}$ small. Then, if $(\bar{\rho}, \bar{p}) \rightarrow (\bar{\lambda}, \bar{\mu})$ we obtain, after differentiating (3.8), (3.9) and using Taylor's Theorem:

$$|\lambda_r(r) - \bar{\lambda}_r(r)| \leq \frac{1}{2} \left| \frac{1 - R'_0(r)}{r - R_0(r)} - \frac{1 - \bar{R}'_0(r)}{r - \bar{R}_0(r)} \right|, \quad a.e. \quad r \in (0, \infty)$$

$$|\mu_r(r) - \bar{\mu}_r(r)| \leq 4\pi \left| \frac{r^2 p(r)}{r - R_0(r)} - \frac{r^2 \bar{p}(r)}{r - \bar{R}_0(r)} \right| + \frac{1}{2} \left| \frac{R_0(r)}{(r - R_0(r))r} - \frac{\bar{R}_0(r)}{(r - \bar{R}_0(r))r} \right|$$

whence:

$$\begin{aligned} \int_0^\infty |\lambda_r(r) - \bar{\lambda}_r(r)| dr &\leq C \int_{\delta_0}^\infty \frac{|R_0(r) - \bar{R}_0(r)|}{r^2} dr + C \int_{\delta_0}^\infty |\rho(r) - \bar{\rho}(r)| dr \\ \int_0^\infty |\mu_r(r) - \bar{\mu}_r(r)| dr &\leq C \int_{\delta_0}^\infty r |p(r) - \bar{p}(r)| dr + C \int_{\delta_0}^\infty p(r) |R_0(r) - \bar{R}_0(r)| dr \\ &\quad + C \int_{\delta_0}^\infty \frac{|R_0(r) - \bar{R}_0(r)|}{r^2} dr \end{aligned}$$

Then, if $\int_0^\infty |\rho(r) - \bar{\rho}(r)| r^2 dr + \int_0^\infty |p(r) - \bar{p}(r)| r^2 dr$ is small we have that $\int_0^\infty |\lambda_r(r) - \bar{\lambda}_r(r)| dr$ and $\int_0^\infty |\mu_r(r) - \bar{\mu}_r(r)| dr$ are small too. Since $(\lambda(\delta_0) - \bar{\lambda}(\delta_0)) = (\mu(\delta_0) - \bar{\mu}(\delta_0)) = 0$ it then follows that (λ, μ) and $(\bar{\lambda}, \bar{\mu})$ are close in $(W^{1,1}(0, L))^2$ for any $L > 0$ and the result follows. ■

One of the technical difficulties that we have to deal with is the fact that the support of the measure f contains turning points. More precisely, there are two admissible velocities $v_1(r, t), v_2(r, t)$ if $r > y_0(-t)$ and no admissible velocities if $r < y_0(-t)$. In a neighbourhood of $r = y_0(-t)$ quantities like ρ and p (and then μ_r, λ_t) are unbounded. Due to this it is not clear in which sense a measure f is a solution of (2.13) unless some continuity assumptions are made in some of the functions appearing in (2.13). These continuity assumptions will provide a relation between the motion of the turning point $r = y_0(-t)$, and the functions $v_1(r, t), v_2(r, t), B_1(r, t), B_2(r, t)$. The precise continuity assumptions needed to give a meaning to the solutions of (2.13) are studied in Lemmas 3 and 7.

Lemma 3 *Let \mathcal{Z}_{δ_0} be as in Lemma 2. Suppose that $\rho, p \in C([0, T]; \mathcal{Z}_{\delta_0})$ for $T < \infty$ and some $\delta_0 > 0$. Let λ, μ be as in (3.8), (3.9). Let us define a new variable \bar{v} by means of $v = \bar{v}e^{-\lambda}$. Let us assume also that the function*

$$\Psi(t, r, \bar{v}) = \left[-e^{\mu-2\lambda} \bar{v}^2 \rho + \left(\bar{v}^2 e^{-2\lambda} + \frac{1}{r^2} \right) p \right] \quad (3.11)$$

is continuous in a set $S \subset [0, T] \times (0, \infty) \times (-\infty, \infty)$. Suppose that $\varphi = \varphi(t, r, \bar{v}, F) \in C_0^1([0, T] \times [0, \infty) \times (-\infty, \infty) \times (0, \infty))$. Then the function $\Delta(t, r, \bar{v}, F)$ defined by means of:

$$\Delta(t, r, \bar{v}, F) = \partial_t \varphi(t, r, \bar{v}, F) + \partial_r \left(e^{\mu-2\lambda} \frac{\bar{v}}{\bar{E}} \varphi \right) - \partial_{\bar{v}} \left(\left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\bar{E}} + e^\mu \mu_r \bar{E} - e^\mu \frac{1}{r^3 \bar{E}} \right) \varphi \right) \quad (3.12)$$

is continuous in $S \subset [0, T] \times [0, \infty) \times (-\infty, \infty) \times [0, \infty)$.

Proof. Notice that $\tilde{E} = \sqrt{\bar{v}^2 e^{-2\lambda} + \frac{1}{r^2}}$. The function $\partial_t \varphi$ is continuous. Due to Lemma 2 λ and μ are also continuous in $(t, r) \in [0, T] \times (0, \infty)$. Then, we only need to check the continuity of

$$\partial_r \left(e^{\mu-2\lambda} \frac{\bar{v}}{\tilde{E}} \varphi \right) - \partial_{\bar{v}} \left(\left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\tilde{E}} + e^\mu \mu_r \tilde{E} - e^\mu \frac{1}{r^3 \tilde{E}} \right) \varphi \right)$$

Notice that, due to the differentiability of φ and the continuity of λ , μ we just need to prove the continuity of the functions:

$$\Delta_1^*(t, r, \bar{v}, F) = \left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\tilde{E}} + e^\mu \mu_r \tilde{E} \right) \quad (3.13)$$

$$\Delta_2^*(t, r, \bar{v}, F) = \partial_r \left(e^{\mu-2\lambda} \frac{\bar{v}}{\tilde{E}} \right) - \partial_{\bar{v}} \left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\tilde{E}} + e^\mu \mu_r \tilde{E} \right) \quad (3.14)$$

The continuity of Δ_1^* in $\{r > 0\}$ is equivalent to the continuity of $(-\lambda_r e^{\mu-2\lambda} \bar{v}^2 + \mu_r (\bar{v}^2 e^{-2\lambda} + \frac{1}{r^2}))$ in the same region. Using (2.5), (2.6) we can rewrite this function as:

$$\frac{1}{2r} \left(-e^{\mu-2\lambda} \bar{v}^2 [(8\pi r^2 \rho - 1) e^{2\lambda} + 1] + \left(\bar{v}^2 e^{-2\lambda} + \frac{1}{r^2} \right) [(8\pi r^2 p + 1) e^{2\lambda} - 1] \right)$$

Since the functions λ and μ are continuous, we just need to check the continuity in $\{r > 0\}$ of

$$8\pi r^2 \left[-e^{\mu-2\lambda} \bar{v}^2 \rho + \left(\bar{v}^2 e^{-2\lambda} + \frac{1}{r^2} \right) p \right] = 8\pi r^2 \Psi(t, r, \bar{v})$$

and due to the continuity of Ψ it then follows that Δ_1^* is continuous.

On the other hand, expanding the derivatives in (3.14) we can see that the continuity of Δ_2^* is equivalent to the continuity of the function:

$$\frac{\bar{v} e^{\mu-2\lambda}}{\tilde{E}} (\mu_r - 2\lambda_r) + e^{\mu-2\lambda} \frac{\bar{v}^3 e^{-2\lambda} \lambda_r}{\tilde{E}^3} + \frac{2\lambda_r e^{\mu-2\lambda} \bar{v}}{\tilde{E}} - \frac{\lambda_r e^{\mu-2\lambda} \bar{v}^3 e^{-2\lambda}}{\tilde{E}^3} - \frac{e^\mu \mu_r e^{-2\lambda} \bar{v}}{\tilde{E}}$$

which turns out to be identically zero. This concludes the proof of the Lemma. \blacksquare

Remark 4 The continuity condition for $\Psi(t, r, \bar{v})$ in (3.11) will be satisfied for the solutions obtained in this paper using (2.9), (2.10).

Remark 5 The assumption that the function Ψ is continuous in the set S is a very strong constraint about the shape of this set. This assumption gives information about the points of the support of S where the coordinate r reaches its minimum. Heuristically, these are the points where "shell crossing" takes place. Notice that we cannot expect the functions Ψ and Δ to be continuous in any neighbourhood of one of such points. However, the functions Ψ and Δ restricted to the set S can be continuous if this set is chosen in a suitable way.

Since the functions p and ρ as well as the fields λ, μ depend on S the continuity condition yields information about the possible geometry of this set near the points where r is minimum. As indicated before Lemma 3, the continuity of the function Δ will be needed in order to define measure-valued solutions supported in S for (2.13), (2.14).

We need to be able to define integrals of measures supported on sets $S \subset [T_1, T_2] \times [0, \infty) \times (-\infty, \infty) \times [0, \infty)$ with $-\infty < T_1 < T_2 < \infty$.

Definition 6 Suppose that f is a Radon measure valued function $f = f(t, r, v, F) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+))$, $-\infty < T_1 < T_2 < \infty$. Let $S \subset [T_1, T_2] \times [0, \infty) \times (-\infty, \infty) \times [0, \infty)$ the support of f . Suppose that $\psi = \psi(t, r, v, F) \in C_0^1(S)$. Let us denote as $\bar{\psi}$ any function $\bar{\psi} \in C_0^1([T_1, T_2], [0, \infty) \times (-\infty, \infty) \times (0, \infty))$ such that $\bar{\psi}(t, r, v, F) = \psi(t, r, v, F)$ for any $(t, r, v, F) \in S$. We define the integral $\int \int_S f \psi dr dv dF dt$ as:

$$\int \int_S f \psi dr dv dF dt = \int_{T_1}^{T_2} \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} f \bar{\psi} dr dv dF dt \quad (3.15)$$

The existence of at least one extension $\bar{\psi}$ of the function ψ as indicated in Definition 6 follows from standard analysis results. We now prove that the Definition 6 is independent of the extension $\bar{\psi}$ used, i.e. the function ψ , which is defined in S , characterizes uniquely the value of $\int \int_S f \psi$. This is proved in the following Lemma.

Lemma 7 Suppose that $\bar{\psi}_1, \bar{\psi}_2$ are two extensions of the function ψ as stated in Definition 6. Then:

$$\int_{T_1}^{T_2} \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} f \bar{\psi}_1 dr dv dF dt = \int_{T_1}^{T_2} \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} f \bar{\psi}_2 dr dv dF dt$$

Proof. Let $\varepsilon > 0$. Suppose that $B_R(0)$ is a large ball in $(\mathbb{R})^4$ containing the support of the functions $\bar{\psi}_1, \bar{\psi}_2$. Using the continuity of these functions, as well as the compactness of the set $S \cap \overline{B_R(0)}$ it follows that there exist a finite family of balls $B_\delta(\xi_j) \subset (\mathbb{R})^4$, $j = 1, \dots, N_\varepsilon$, $\xi_j = (t_j, r_j, v_j, F_j)$ with $\delta > 0$ depending on ε , such that $S \cap \overline{B_R(0)} \subset \bigcup_{j=1}^{N_\varepsilon} B_\delta(\xi_j)$ and $|\bar{\psi}_\ell(y_1) - \bar{\psi}_\ell(y_2)| < \varepsilon$ for $\ell = 1, 2$, $y_1, y_2 \in B_\delta(\xi_j)$. Moreover, we can assume also that $B_\delta(\xi_j) \cap S \neq \emptyset$ for any $j = 1, \dots, N_\varepsilon$. We construct a partition of the unity $\{\zeta_k\}_{k=1}^{N_\varepsilon}$ such that $\sum_{j=1}^{N_\varepsilon} \zeta_j = 1$ in $\bigcup_{k=1}^{N_\varepsilon} \overline{B_\delta(\xi_k)}$, $\zeta_j \geq 0$ and $\sum_{j=1}^{N_\varepsilon} \zeta_j = 0$ at $\xi = (t, r, v, F)$ if $\text{dist}(\xi, \bigcup_{k=1}^{N_\varepsilon} \overline{B_\delta(\xi_k)}) \geq 1$. Notice that, since $B_\delta(\xi_j) \cap S \neq \emptyset$ and $\bar{\psi}_1(y) = \bar{\psi}_2(y)$ if $y \in S$ we have that $|\bar{\psi}_1(y) - \bar{\psi}_2(y)| < 2\varepsilon$ for any $y \in B_\delta(\xi_j)$. We then

have:

$$\begin{aligned}
& \left| \int_{T_1}^{T_2} \int_{(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \cap B_R(0)} f(\bar{\psi}_1 - \bar{\psi}_2) dr dv dF dt \right| \\
&= \left| \sum_{j=1}^{N_\varepsilon} \int_{T_1}^{T_2} \int_{(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \cap B_R(0)} f(\bar{\psi}_1 - \bar{\psi}_2) \zeta_j dr dv dF dt \right| \\
&\leq \varepsilon \sum_{j=1}^{N_\varepsilon} \int_{T_1}^{T_2} \int_{(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \cap B_R(0)} f \zeta_j dr dv dF dt \\
&\leq \varepsilon \int_{T_1}^{T_2} \int_{(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+) \cap B_R(0)} f dr dv dF dt
\end{aligned}$$

and since ε is arbitrarily small the result follows. ■

We can now define our concept of measure-valued solution for the spherically symmetric Einstein-Vlasov system.

Definition 8 Given a Radon measure $f = f(t, r, v, F) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+))$, $-\infty < T_1 < T_2 < \infty$ suppose that ρ, p defined by means of (2.9), (2.10) are in $C([T_1, T_2]; \mathcal{Z}_{\delta_0})$ for some $\delta_0 > 0$, and that the functions λ, μ are given by (3.8), (3.9) for each $t \in [0, T]$. Suppose also that for any set of the form $\mathcal{U} = \{(t, r, v, F) : (t, r) \in A, v \in \mathbb{R}, F \in \mathbb{R}_+\}$ with A measurable in $[T_1, T_2] \times \mathbb{R}_+$ we have $\int_{\mathcal{U}} F f < \infty$. Let us denote as $\tilde{f}(t, r, \bar{v}, F)$ the measure defined by means of:

$$\tilde{f}(t, r, \bar{v}, F) = f(t, r, \bar{v}e^{-\lambda}, F) \quad (3.16)$$

and let us denote the support of \tilde{f} as S . We will say that f is a solution of (2.3)-(2.10) in the sense of measures in the interval $t \in [T_1, T_2]$ if the function $\Psi(t, r, \bar{v})$ defined in (3.11) is continuous in $S \subset [T_1, T_2] \times (0, \infty) \times (-\infty, \infty) \times [0, \infty)$ and for any test function $\varphi = \varphi(t, r, \bar{v}, F) \in C_0^1(T_1, T_2, \times [0, \infty) \times (-\infty, \infty) \times [0, \infty))$ the following identity holds:

$$\int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} \tilde{f}(T_1, r, \bar{v}, F) \varphi(T_1, r, \bar{v}, F) dr d\bar{v} dF + \int_S \tilde{f}(t, r, \bar{v}, F) \Delta(t, r, \bar{v}e^{-\lambda t}, F) dr d\bar{v} dF dt = 0 \quad (3.17)$$

where $\tilde{E} = \sqrt{\bar{v}^2 e^{-2\lambda} + \frac{1}{r^2}}$, $\Delta(r, v, F, t)$ is as in (3.12) and the integral (3.17) is understood in the sense of Definition 6.

Remark 9 Changes of variables in measures are defined, in the usual manner, by means of the change of variables over the test function, i.e., the measure defined in (3.16) must be understood as:

$$\int \tilde{f}(t, r, \bar{v}, F) \varphi(t, r, \bar{v}, F) dt dr d\bar{v} dF = \int f(t, r, v, F) \tilde{\varphi}(t, r, v, F) dt dr dv dF$$

for any test function φ where $\tilde{\varphi}(t, r, v, F) = \varphi(t, r, ve^\lambda, F) e^\lambda$.

Remark 10 Note that the second integral in (3.17) is well defined due to Lemmas 3 and 6.

Remark 11 Notice that this definition excludes the possibility of the support of f reaching $r = 0$.

Remark 12 The reason to use the variable $v = \bar{v}e^{-\lambda}$, in (3.16) as well as in (3.17) is because with this change of variables the regularity assumptions required for the fields λ and μ are smaller. This will become apparent in Subsection 5.2, because the use of the variable \bar{v} in Definition 8 is equivalent to the change of variables (5.20) there. This change of variables allows to eliminate a term λ_τ in the equations for the evolution of the particle densities.

We also need to define weak solutions for (2.5)-(2.8), (2.16), (2.17).

Definition 13 Given a Radon measure $\zeta = \zeta(t, r, v) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R}))$, $-\infty < T_1 < T_2 < \infty$, suppose that ρ, p defined by means of (2.17) are in $C([T_1, T_2]; \mathcal{Z}_{\delta_0})$ for some $\delta_0 > 0$, and that the functions λ, μ are given by (3.8), (3.9) for each $t \in [T_1, T_2]$. Let us denote as $\tilde{\zeta}(t, r, \bar{v})$ the measure defined by means of:

$$\tilde{\zeta}(t, r, \bar{v}) = \zeta(t, r, \bar{v}e^{-\lambda}) \quad , \quad v = \bar{v}e^{-\lambda} \quad (3.18)$$

Let us denote the support of ζ as S . We will say that ζ is a solution of (2.5)-(2.8), (2.16), (2.17) in the sense of measures in the interval $t \in [T_1, T_2]$ if the function $\Psi(t, r, \bar{v})$ defined in (3.11) is continuous in $S \subset [T_1, T_2] \times (0, \infty) \times (-\infty, \infty)$ and for any test function $\bar{\varphi} = \bar{\varphi}(t, r, \bar{v}) \in C_0([T_1, T_2], [0, \infty) \times (-\infty, \infty))$ the following identity holds:

$$\begin{aligned} & \int_{\mathbb{R}_+ \times \mathbb{R}} \tilde{\zeta}(T_1, r, \bar{v}) \bar{\varphi}(0, r, \bar{v}) dr d\bar{v} + \int \int_S \tilde{\zeta}(t, r, \bar{v}) \bar{\Delta}(t, r, \bar{v}) dr d\bar{v} dt \\ & = 0 \end{aligned} \quad (3.19)$$

where:

$$\bar{\Delta}(t, r, \bar{v}) = \partial_t \bar{\varphi}(t, r, \bar{v}) + \partial_r \left(e^{\mu-2\lambda} \frac{\bar{v}}{\bar{E}} \bar{\varphi} \right) - \partial_{\bar{v}} \left(\left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\bar{E}} + e^\mu \mu_r \bar{E} - e^\mu \frac{1}{r^3 \bar{E}} \right) \bar{\varphi} \right) \quad (3.20)$$

Remark 14 The measure $\tilde{\zeta}$ must be understood in a manner similar to the one in Remark 9, with minor changes due to the fact that we consider functions and measures in a space with one variable less.

Remark 15 Lemma 3 applied to the test function φ , which is independent of F , implies that the function Δ is continuous and therefore, the second integral in (3.19) is well defined.

Remark 16 A definition of weak solutions for the one-dimensional Vlasov-Poisson system has been given in [14], [19]. The definition in that paper allows

to give a meaning to the solutions of that system in the cases in which the density f belongs to some particular class of measures, including Dirac masses. It is not obvious if it is possible to adapt the definition used in [14], [19] to the Einstein-Vlasov system, and if the resulting definition would be equivalent to the concept of weak solution introduced in the Definitions 8, 13. Notice that we do not try to define a concept of solution for densities f containing Dirac masses, but just for measures f supported in surfaces in the space (r, v, F) or measures ζ supported in a curve in the plane (r, v) but having possible turning points. The assumption about the continuity of the function Ψ in (3.11) determines the motion of the turning points.

It is relevant to characterize the relation between the measure-valued solutions of (2.3)-(2.10) and the measure-valued solutions of (2.5)-(2.8), (2.16), (2.17).

Proposition 17 *Suppose that $f = f(t, r, v, F) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+))$, $-\infty < T_1 < T_2 < \infty$, is a solution of (2.3)-(2.10) in the sense of measures in the interval $t \in [T_1, T_2]$ with initial datum $f_0(r, v, F) = f(T_1, r, v, F)$ (cf. Definition 8). We define $\zeta = \zeta(t, r, v) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R}))$ by means of (2.15). Then ζ is a solution of (2.5)-(2.8), (2.16), (2.17) in the sense of measures in the interval $t \in [T_1, T_2]$ with initial datum $\zeta(T_1, r, v) = \int_0^\infty F f_0(r, v, F) dF$ (cf. Definition 13).*

Proof. Notice that, due to Definition 8 we have that the functions ρ , p defined by means of (2.17) are in $C([0, T]; \mathcal{Z}_{\delta_0})$ for some $\delta_0 > 0$. Then the measure ζ given by (2.15) is well defined. The result then follows taking as test function in (3.17) a sequence of test functions $\varphi_\varepsilon(t, r, \bar{v}, F) = \bar{\varphi}(t, r, \bar{v}) \zeta_\varepsilon(F)$ with $\bar{\varphi}(t, r, \bar{v}) \in C_0([0, \infty) \times [0, \infty) \times (-\infty, \infty))$ and ζ_ε satisfying $0 \leq \zeta_\varepsilon(F) \leq F$, $\zeta_\varepsilon(F) \leq C_\varepsilon < \infty$, $\lim_{\varepsilon \rightarrow 0} \zeta_\varepsilon(F) = F$. Since the integrals with the form $\int \int_{\mathcal{U}} F f$ are finite, due to Definition 8. we can take the limit $\varepsilon \rightarrow 0$, using Lebesgue's Theorem to obtain the result. ■

Reciprocally, given ζ solution of (2.5)-(2.8), (2.16), (2.17) in the sense of the Definition 13, we can obtain a large class of measures f which solve (2.3)-(2.10) in the sense of Definition 8. The key idea underlying the proof of the following result is that the angular momentum is constant along the characteristics associated to the equation (2.3).

Proposition 18 *Suppose that $\zeta = \zeta(t, r, v) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R}))$, $-\infty < T_1 < T_2 < \infty$, is a solution of (2.5)-(2.8), (2.16), (2.17) in the sense of Definition 13. Let us assume that $\xi \in \mathcal{M}_+(0, \infty)$ is a compactly supported measure. We define measures $f = f(t, r, v, F) \in C([T_1, T_2], \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+))$ by means of:*

$$f(t, r, v, F) = \zeta(t, r, v) \xi(F) \quad (3.21)$$

Then, f is a solution of (2.3)-(2.10) in the sense of Definition 8.

Proof. We choose a test function $\varphi = \varphi(t, r, \bar{v}, F) \in C_0^1([T_1, T_2] \times [0, \infty) \times (-\infty, \infty) \times [0, \infty))$ (cf. Definition 8). Using (3.21) we can rewrite the right-hand side of (3.17) as:

$$\begin{aligned} J &= \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} \tilde{\zeta}(T_1, r, \bar{v}) \xi(F) \varphi(T_1, r, \bar{v}, F) dr d\bar{v} dF \\ &\quad + \int_S \int \tilde{\zeta}(t, r, \bar{v}) \xi(F) \Delta(t, r, \bar{v}e^{-\lambda t}, F) dr d\bar{v} dF dt \end{aligned} \quad (3.22)$$

where $\tilde{\zeta}$ is as in (3.18). We define:

$$\bar{\varphi}(t, r, \bar{v}) = \int \xi(F) \varphi(t, r, \bar{v}, F) dF \quad (3.23)$$

Using (3.12) it then follows that the function $\bar{\Delta}$ associated to $\bar{\varphi}$ is given by:

$$\bar{\Delta}(t, r, \bar{v}) = \int \xi(F) \Delta(t, r, \bar{v}, F) dF \quad (3.24)$$

Using (3.23), (3.24) we obtain that J in (3.22) is given by:

$$J = \int_{\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+} \tilde{\zeta}(T_1, r, \bar{v}) \bar{\varphi}(T_1, r, \bar{v}) dr d\bar{v} + \int_S \int \tilde{\zeta}(t, r, \bar{v}) \bar{\Delta}(t, r, \bar{v}e^{-\lambda t}, F) dr d\bar{v} dt$$

Therefore, using (3.19) we obtain $J = 0$ whence the result follows. ■

3.3 Main results.

The main theorem that will be proved in this paper is the following:

Theorem 19 *There exists $t_0 < 0$ and a measure $\zeta = \zeta(t, r, v) \in C([t_0, 0] : \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R}))$ supported in two curves $\gamma_1(t)$, $\gamma_2(t)$ which can be parametrized in the form:*

$$\begin{aligned} \gamma_1(t) &= \{v = v_1(t, r) \text{ , } r \geq y_0(-t) \text{ , } t_0 \leq t < 0\} \\ \gamma_2(t) &= \{v = v_2(t, r) \text{ , } y_0(-t) \leq r \leq R_+(t) \text{ , } t_0 \leq t < 0\} \end{aligned}$$

for some $y_0 > 0$ and suitable functions $v_1 \in C^1\left(\bigcup_{t \in (t_0, 0)} [(y_0(-t), \infty) \times \{t\}]\right)$, $v_2 \in C^1\left(\bigcup_{t \in (t_0, 0)} [(y_0(-t), R_+(t)) \times \{t\}]\right)$, $R_+ \in C^1(t_0, 0)$ and such that, the functions ρ , p given by (2.17) satisfy ρ , $p \in L_{\text{loc}}^\infty((y_0(-t), \infty) \times (t_0, 0))$ and:

$$\begin{aligned} 0 &\leq \limsup_{r \rightarrow (y_0(-t))^+} \sqrt{r - y_0(-t)} \rho(t, r) \leq C(T) < \infty \\ 0 &\leq \limsup_{r \rightarrow (y_0(-t))^+} \sqrt{r - y_0(-t)} p(t, r) \leq C(T) < \infty \end{aligned}$$

for any $t_0 < T < 0$. Moreover, ζ solves (2.5)-(2.8), (2.16), (2.17) in the sense of the Definition 13.

The distribution $\zeta(r, v, t)$ has the following asymptotics:

$$\zeta(t, r, v) = \frac{r\chi_{(0, R_{\max})}(r)}{12\pi^2} \delta(v) + O\left((-t)^b \exp(-ar)\right) \quad \text{as } t \rightarrow 0 \quad (3.25)$$

where $R_{\max} > 0$ is a fixed number, $a > 0$, $b > 0$ and $\chi_{(0, R_{\max})}(r)$ is the characteristic function supported in the interval $(0, R_{\max})$. The asymptotics (3.25) must be understood in the sense of distributions, i.e. after multiplying by a test function.

Moreover, the metric (2.1) defined using these functions behaves asymptotically as:

$$ds^2 = -e^{2\mu(t, r)} dt^2 + e^{2\lambda(t, r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$

where:

$$e^{2\lambda(t, r)} \rightarrow 1, \quad e^{2\mu(t, r)} \rightarrow e^{2\mu_\infty} \quad \text{as } r \rightarrow \infty, \quad t \in (t_0, 0) \quad (3.26)$$

for some suitable $\mu_\infty > 0$.

Combining Proposition 18 and Theorem 19 we obtain the following result:

Theorem 20 *There exist $t_0 < 0$ and infinitely many measures $f = f(t, r, v, F) \in C((t_0, 0), \mathcal{M}_+(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+))$ supported in the surfaces $\gamma_k(t) \times \mathbb{R}_+$, $k = 1, 2$, with $\gamma_1(t)$, $\gamma_2(t)$ as in Theorem 19, such that f solves (2.3)-(2.10) in the interval $t \in (t_0, 0)$ in the sense of Definition 8. The measure f satisfies $\int_{\mathbb{R}_+} f dF = \zeta$, with ζ as in Theorem 19. The measure ζ satisfies (3.25). The metric (2.1) is asymptotically flat as $r \rightarrow \infty$ in the sense of (3.26).*

Remark 21 *It is relevant to remark that the asymptotics (3.25) shows that the particle distribution behaves asymptotically, for times close to the singularity, like a particular type of generalized Einstein clusters whose speed approaches zero for times close to the onset of the singularity. The gravitational field for large values of r , in the region where the metric becomes asymptotically flat, is, for long times, the corresponding one to that generalized Einstein cluster.*

4 MAIN PROPERTIES OF THE SELF-SIMILAR SOLUTION CONSTRUCTED IN [17].

We summarize some of the main properties of the solution of (2.5), (2.6), (2.16), (2.17) constructed in [17]. Most of the results of the next Theorem have been proved in [17] and we just reformulate some specific points in a form that is more convenient in order to obtain the results of this paper.

Theorem 22 *For any $y_0 > 0$ sufficiently small, there exists a solution of (2.5), (2.6), (2.16), (2.17) of the form (3.3), (3.4) where $\Theta(y, V)$ can be written in the form:*

$$\Theta(y, V) = \beta_0 e^{2\sigma} \delta(H - h) \quad , \quad H = \frac{e^U}{y} \sqrt{V^2 y^2 + 1} + y V e^\Lambda, \quad (4.1)$$

where $\beta_0 = \beta_0(y_0) > 0$, $h = \frac{\sqrt{1-y_0^2}}{y_0}$. The fields λ , μ have the self-similar form $\Lambda(y)$, $U(y)$ in (3.3). The curve $\{H = h\} \subset \{(y, V) : y \geq y_0\}$ can be decomposed into two portions:

$$\{H = h\} = \Gamma_1 \cup \Gamma_2, \quad \Gamma_k = \{(y, V) : V = V_1(y), y \geq y_0\}, \quad k = 1, 2 \quad (4.2)$$

with $V_1(y) < V_2(y)$ for $y > y_0$ and $V_1(y_0) = V_2(y_0) = -\frac{1}{\sqrt{1-y_0^2}}$.

Each of the curves Γ_1 , Γ_2 can be parametrized using the parameter σ in (4.1):

$$\begin{aligned} \Gamma_1 &= \{(y, V) : y = \bar{y}_1(\sigma), V = \bar{V}_1(\sigma), \sigma \leq 0\} \\ \Gamma_2 &= \{(y, V) : y = \bar{y}_2(\sigma), V = \bar{V}_2(\sigma), \sigma \geq 0\} \end{aligned}$$

where the functions \bar{y} , \bar{V}_k , $k = 1, 2$ solve the ODEs:

$$\begin{aligned} \frac{d\bar{y}_k}{d\sigma} &= \bar{y}_k + e^{U(\bar{y}_k) - \Lambda(\bar{y}_k)} \frac{\bar{V}_k}{\mathcal{E}_k} \\ \frac{d\bar{V}_k}{d\sigma} &= -\bar{V}_k - \left(\bar{y}_k \Lambda_y(\bar{y}_k) \bar{V}_k + e^{U(\bar{y}_k) - \Lambda(\bar{y}_k)} \mathcal{E}_k U_y(\bar{y}_k) - \frac{e^{U(\bar{y}_k) - \Lambda(\bar{y}_k)}}{\bar{y}_k^2 \mathcal{E}_k} \right) \end{aligned}$$

with:

$$\mathcal{E}_k = \sqrt{\frac{1}{\bar{y}_k^2} + \bar{V}_k^2}, \quad \sigma(y_0; y_0) = 0$$

for $k = 1, 2$.

The measure $\Theta(y, V)$ in (4.1) can be rewritten in the form:

$$\Theta(y, V) = b_1(y) \delta(V - V_1(y)) + b_2(y) \delta(V - V_2(y)) \quad (4.3)$$

where the functions $V_1(\cdot)$, $V_2(\cdot)$ are as in (4.2) and:

$$b_1(y) = \frac{\beta_0 e^{2\sigma}}{\left| \frac{\partial H}{\partial V}(y, V_1(y)) \right|}, \quad b_2(y) = \frac{\beta_0 e^{2\sigma}}{\left| \frac{\partial H}{\partial V}(y, V_2(y)) \right|} \quad (4.4)$$

Moreover, if we write ζ in (3.3) in the form (3.2) we obtain:

$$B_k(t, r) = (-t) b_k(y), \quad v_k(t, r) = \frac{V_k(y)}{(-t)}, \quad k = 1, 2. \quad (4.5)$$

The following asymptotics hold:

$$U(y) = \log\left(\frac{y}{y_0}\right) + \log\left(\sqrt{1-y_0^2}\right) + O\left(\frac{1}{y^\delta}\right) \quad \text{as } y \rightarrow \infty \quad (4.6)$$

$$\Lambda(y) = \log(\sqrt{3}) + O\left(\frac{1}{y^\delta}\right) \quad \text{as } y \rightarrow \infty \quad (4.7)$$

$$V_1(y) = -\frac{2y_0\sqrt{3(1-y_0^2)}}{(1-4y_0^4)y} \left(1 + O\left(\frac{1}{y^\delta}\right)\right) \quad \text{as } y \rightarrow \infty \quad (4.8)$$

$$V_2(y) = -\frac{\sqrt{1-y_0^2}}{\sqrt{3}y_0} \frac{C_1}{y} \left(\frac{y_0}{y}\right)^2 (1 + o(1)) \quad \text{as } y \rightarrow \infty \quad (4.9)$$

$$b_1(y) \sim A_1(y)^{1-\gamma(y_0)} \quad , \quad b_2(y) \sim \frac{y}{12\pi^2} \text{ as } y \rightarrow \infty \quad (4.10)$$

$$A_1 = \frac{\beta_0 (C_2)^2 (y_0)^{\frac{4(1-y_0^2)}{(1-4y_0^2)}-2}}{\left| \frac{\zeta_1 \sqrt{1-y_0^2}}{y_0 \sqrt{(\zeta_1)^2+1}} + \sqrt{3} \right|} \quad , \quad \gamma(y_0) = \frac{4(1-y_0^2)}{(1-4y_0^2)} \quad , \quad \zeta_1 = -\frac{2y_0 \sqrt{3(1-y_0^2)}}{(1-4y_0^2)} \quad (4.11)$$

for some constants $C_1, C_2 \in \mathbb{R}$ and $\delta > 0$, depending on y_0 . Moreover, for any compact $\mathcal{K} \subset (0, \infty)$ and any positive integer m , there exists a constant C depending on \mathcal{K} and m such that:

$$B_1(t, r) = A_1(-t)^{\gamma(y_0)} r^{1-\gamma(y_0)} + O\left((-t)^{\gamma(y_0)+\delta}\right) \quad , \quad B_2(t, r) = \frac{r}{12\pi^2} + O\left((-t)^\delta\right) \text{ as } t \rightarrow 0^- \quad (4.12)$$

$$\left| \frac{\partial^\ell}{\partial r^\ell} \left(v_1(t, r) - \frac{\zeta_1}{r} \right) \right| \leq C(-t)^\delta \quad , \quad \left| \frac{\partial^\ell}{\partial r^\ell} \left(v_2(t, r) + \frac{\sqrt{1-y_0^2}}{\sqrt{3}y_0} \frac{C_1(-t)^2}{r} \left(\frac{y_0}{r} \right)^2 \right) \right| \leq C(-t)^\delta \quad (4.13)$$

$$\left| \frac{\partial^\ell}{\partial r^\ell} \left(\lambda(t, r) - \log(\sqrt{3}) \right) \right| \leq C(-t)^\delta \quad \text{as } t \rightarrow 0 \quad (4.14)$$

$$\left| \frac{\partial^\ell}{\partial r^\ell} \left(\mu(t, r) - \log\left(\frac{1}{(-t)}\right) - \log(r) - \log\left(\frac{\sqrt{1-y_0^2}}{y_0}\right) \right) \right| \leq C(-t)^\delta \quad \text{as } t \rightarrow 0 \quad (4.15)$$

for $\ell = 0, 1, \dots, m$, $r \in \mathcal{K}$ and $|t|$ small.

Proof. We first refer to the specific points of Theorem 22 which have been proved in [17]. The representation formula (4.4) just follows from (4.1) and the definition of the functions $V_1(y)$, $V_2(y)$. The asymptotics (4.10) is a consequence of a representation formula that has been obtained in [17], namely:

$$e^{\sigma_k} = \frac{y}{y_0} Q_k \quad , \quad k = 1, 2 \quad (4.16)$$

as well as the following asymptotics (cf. also [17]):

$$Q_1 \sim C_2 e^{-\frac{2(1-y_0^2)}{(1-4y_0^2)}s} \text{ as } s \rightarrow \infty \quad , \quad Q_2 \rightarrow Q_{2,\infty} = \frac{2\sqrt{y_0}}{3^{\frac{1}{4}}\sqrt{\theta}} \quad , \quad \theta = \frac{16\pi^2\beta_0}{y_0} \quad , \quad \frac{y}{y_0} = e^s \quad (4.17)$$

for some $C_2 \in \mathbb{R}$. Moreover, using (4.1), (4.6)-(4.9) we obtain:

$$\begin{aligned} \frac{\partial H}{\partial V}(y, V_2(y)) &\sim \sqrt{3}y \\ \frac{\partial H}{\partial V}(y, V_1(y)) &\sim \left(\frac{\zeta_1 \sqrt{1-y_0^2}}{y_0 \sqrt{(\zeta_1)^2+1}} + \sqrt{3} \right) y \quad , \quad \zeta_1 = -\frac{2y_0 \sqrt{3(1-y_0^2)}}{(1-4y_0^2)} \end{aligned}$$

as $y \rightarrow \infty$. Combining (4.16), (4.17) we then obtain (4.10). Using (4.5) we then obtain (4.12). Finally, the asymptotics (4.13)-(4.15) is a consequence of (4.6)-(4.9).

We now check that the measure ζ given in (3.3), (3.4) satisfies Definition 13. To this end we rewrite the functions $\Psi(t, r, \bar{v})$, $\Delta(t, r, \bar{v})$ in (3.11), (3.20) respectively using the self-similar variables (3.4). Then:

$$\begin{aligned} \Psi(t, r, \bar{v}) = & \frac{\pi}{y^2} \frac{1}{(-t)^4} \left[-e^{U(y)} V^2 \int_{-\infty}^{\infty} \sqrt{V^2 + \frac{1}{y^2}} \Theta(y, V) dV \right. \\ & \left. + \left(V^2 + \frac{1}{y^2} \right) \int_{-\infty}^{\infty} \frac{V^2}{\sqrt{V^2 + \frac{1}{y^2}}} \Theta(y, V) dV \right] \end{aligned} \quad (4.18)$$

On the other hand, suppose that we define Φ by means of:

$$\varphi(t, r, \bar{v}) = \Phi(\tau, y, V)$$

where y, V are as in (3.3) and $\tau = \log\left(\frac{1}{(-t)}\right)$. We can then compute $\Delta(t, r, \bar{v})$ defined in (3.12):

$$\begin{aligned} \Delta(t, r, \bar{v}) = & \frac{1}{(-t)} [\Phi_\tau + y\Phi_y - V\Phi_V - y\Lambda_y V\Phi_V] \\ & + \left[\frac{1}{(-t)} (U_y - 2\Lambda_y) e^{U-\Lambda} \frac{V}{\sqrt{V^2 + \frac{1}{y^2}}} + \frac{e^{U-\Lambda}}{(-t)} \frac{V}{\sqrt{(V^2 + \frac{1}{y^2})^3}} \left(\Lambda_y V^2 + \frac{1}{y^3} \right) \right] \Phi \\ & + \frac{1}{(-t)} e^{U-\Lambda} \frac{V}{\sqrt{V^2 + \frac{1}{y^2}}} [\Phi_y - \Lambda_y V\Phi_V] \\ & - \frac{\Phi e^{U-\Lambda}}{(-t)} \left[-\frac{2\Lambda_y V}{\sqrt{V^2 + \frac{1}{y^2}}} + \frac{\Lambda_y V^3}{\sqrt{(V^2 + \frac{1}{y^2})^3}} + \frac{U_y V}{\sqrt{V^2 + \frac{1}{y^2}}} + \frac{V}{y^3 \sqrt{(V^2 + \frac{1}{y^2})^3}} \right] \\ & - \frac{e^{U-\Lambda}}{(-t)} \left(-\frac{\Lambda_y V^2}{\sqrt{V^2 + \frac{1}{y^2}}} + U_y \sqrt{V^2 + \frac{1}{y^2}} - \frac{1}{y^3 \sqrt{V^2 + \frac{1}{y^2}}} \right) \Phi_V \end{aligned} \quad (4.19)$$

Using (4.1) and (4.18) we can check that $\Psi(t, r, \bar{v})$ is a continuous function on the support of S . Indeed, using (4.3) we obtain:

$$\begin{aligned} \Psi(t, r, \bar{v}) = & \frac{\pi}{y^2} \frac{1}{(-t)^4} \left[-e^{U(y)} V^2 \sum_{k=1}^2 b_k(y) \sqrt{(V_k(y))^2 + \frac{1}{y^2}} \right. \\ & \left. + \left(V^2 + \frac{1}{y^2} \right) \sum_{k=1}^2 b_k(y) \frac{(V_k(y))^2}{\sqrt{(V_k(y))^2 + \frac{1}{y^2}}} \right] \end{aligned}$$

and now using (4.4) we obtain that Ψ is continuous for $y > y_0$. It only remains to check the continuity of Ψ at the point $(y, V) = (y_0, V_0)$, with $V_0 = -\frac{1}{\sqrt{1-y_0^2}}$. This can be seen, using the asymptotics of the functions $U(y)$, $V_k(y)$ and $b_k(y)$, in a manner similar to the proof of Proposition 5 of [17]. ■

5 DUST-LIKE SOLUTIONS: DIFFERENTIAL EQUATIONS AND FUNCTION SPACES.

5.1 Some definitions.

As indicated in Section 3, the solution that we will construct agrees with the self-similar solution described in Section 4 for $r \leq R_+(t)$. On the other hand, the solution that we will construct has the form (3.2) with $B_2(t, r) = 0$ for $r > R_+(t)$ and therefore it is not self-similar for $r \geq R_+(t)$. We now derive the equations that must be satisfied by the functions $R_+(t)$, $v_1(t, r)$, $B_1(t, r)$ in order to obtain a solution of (2.5), (2.6), (2.16), (2.17). We also prove the existence of some auxiliary functions which will be needed in the following.

We first observe that the asymptotics (4.15) suggests to introduce a new time scale in order to describe the region where r is of order one. We define:

$$\tau = \log \left(\frac{1}{(-t)} \right) \quad (5.1)$$

as well as:

$$\mu(t, r) = \log \left(\frac{1}{(-t)} \right) + \bar{\mu}(t, r) \quad (5.2)$$

It is also convenient to define an auxiliary function \hat{U} by means of (cf. (4.6)):

$$U(y) = \log(y) + \hat{U}(y) \quad (5.3)$$

It is interesting to remark that for the solution constructed in this paper, t will be the proper time for a particle fixed at the center $r = 0$, while τ is the proper time of a particle at rest at $r = \infty$.

We first describe the behaviour of the function $R_+(t) = r_+(\tau)$. The point $(r, v) = (R_+(t), v_2(t, R_+(t)))$ is a point where there is a discontinuity of the density B_2 associated to the measure ζ . In order to obtain a weak solution of (2.16) the point $(R_+(t), v_2(t, R_+(t)))$ must move along characteristics. Given that the fields λ , μ are continuous at the point $r = R_+(t)$ we can use the values of the self-similar fields in (3.3), (3.4) (cf. also Theorem 22). In order to fix the form of this function we need to impose an additional condition. We will assume that:

$$\lim_{t \rightarrow 0^+} R_+(t) = R_{\max} \quad (5.4)$$

for some $R_{\max} > 0$. Given the form of the characteristic curves associated to (2.16) we define a function $R_+(t)$ by means of the ODE problem:

$$\frac{dr_+(\tau)}{d\tau} = \frac{\exp\left(\hat{U}(r_+(\tau)e^\tau) - \Lambda(r_+(\tau)e^\tau)\right) V_2(r_+(\tau)e^\tau) r_+(\tau)}{\sqrt{(V_2(r_+(\tau)e^\tau))^2 + \frac{e^{2\tau}}{(r_+(\tau))^2}}} \quad (5.5)$$

$$\lim_{\tau \rightarrow \infty} r_+(\tau) = R_{\max} \quad (5.6)$$

where the function $V_2(y)$ is as in Theorem 22. We then define $R_+(t)$ by means of $R_+(t) = r_+(\tau)$.

We need to define in a precise manner some of the functions needed for the fixed point argument. As a first step we construct the function $r_+(\tau)$ which solves (5.5), (5.6).

Due to (4.9) it follows that the right-hand side of (5.5) behaves like $Ce^{-2\tau}$ as $\tau \rightarrow \infty$ for some suitable $C \in \mathbb{R}$ if $r_+(\tau) \rightarrow R_{\max}$ as $\tau \rightarrow \infty$. This will imply the existence of at least one solution of (5.5), (5.4). Similar estimates might be derived for the derivatives of $r_+(\tau)$. More precisely:

Proposition 23 *For any $R_{\max} > 0$, there exists $t_0 < 0$ such that there exists a unique solution of (5.5), (5.4) defined for $-\log(t_0) \leq \tau < 0$. Moreover, we have:*

$$|r_+(\tau) - R_{\max}| + \left| \frac{dr_+(\tau)}{d\tau} \right| \leq Ce^{-4\tau} \quad (5.7)$$

for any $\tau \geq -\log(t_0)$ where C is a constant that depends in general on y_0 , but not on τ . We will write $R_+(t) = r_+(\tau)$.

Proof. The asymptotics (4.6)-(4.9) as well as (4.12)-(4.15) imply that (5.5) can be written in the form:

$$\frac{dr_+(\tau)}{d\tau} = e^{-4\tau} F(\tau, r_+(\tau)) \quad (5.8)$$

where F is a bounded function as well as its first derivatives if $r_+(\tau) \in [\frac{R_{\max}}{2}, R_{\max}]$ and $\tau \geq 0$. We can then reformulate (5.5), (5.4) as the integral equation:

$$r_+(\tau) = R_{\max} - \int_{\tau}^{\infty} e^{-4s} F(s, r_+(s)) ds \quad (5.9)$$

A fixed point argument then shows that there exists a unique solution of (5.9) defined for $-\log(t_0) \leq \tau < 0$ if $|t_0|$ is sufficiently small. The estimates for the derivatives are then obtained using (5.8). ■

5.2 Evolution equations satisfied by $v_k(t, r)$, $B_k(t, r)$, $k = 1, 2$.

In order to obtain solutions of (2.5), (2.6), (2.16), (2.17) with the form (3.2) in the region where $r > R_+(t)$ we need to derive the evolution equations satisfied

by the functions v_k , B_k for $k = 1, 2$. Since in the region $r > R_+(t)$ we assume that $B_2(t, r) = 0$ we need to obtain only the evolution equations for v_1 , B_1 . To this end we impose that ζ given in (3.2) solves (2.16) in the sense of distributions. We derive formally in this subsection the system of differential equations that must be satisfied by $B_1(t, r)$, $v_1(t, r)$ and we will check later that the resulting measure ζ satisfies (2.5)-(2.8), (2.16), (2.17) in the sense of Definition 13. Notice that, by assumption:

$$\zeta(t, r, v) = B_1(t, r) \delta(v - v_1(t, r)) \quad , \quad r > R_+(t) \quad (5.10)$$

Then, the following identities hold in the sense of distributions:

$$\begin{aligned} \partial_\alpha \zeta &= (\partial_\alpha B_1) \delta(v - v_1(t, r)) - B_1 \partial_\alpha v_1(t, r) \delta'(v - v_1(t, r)) \quad , \quad \alpha = t, r \\ \partial_v \zeta &= B_1 \delta'(v - v_1(t, r)) \end{aligned}$$

We will use now the following distributional identity:

$$\begin{aligned} A(t, r, v) \delta'(v - v_1(t, r)) &= A(t, r, v_1(t, r)) \delta'(v - v_1(t, r)) \\ &\quad - A_v(t, r, v_1(t, r)) \delta(v - v_1(t, r)) \end{aligned}$$

Then, using also (2.16) and (5.2), (5.1):

$$\partial_\tau v_1(t, r) + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r v_1 + \left(\lambda_\tau v_1 + e^{\bar{\mu}-\lambda} \mu_r \tilde{E} - e^{\bar{\mu}-\lambda} \frac{1}{r^3 \tilde{E}} \right) = 0 \quad \text{if } B_1 \neq 0 \quad (5.11)$$

$$\partial_\tau B_1 + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r B_1 + \left(e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} (\partial_r v_1) + \lambda_\tau v_1 + e^{\bar{\mu}-\lambda} \mu_r \tilde{E} - e^{\bar{\mu}-\lambda} \frac{1}{r^3 \tilde{E}} \right)_v (t, r, v_1) B_1 = 0 \quad (5.12)$$

with \tilde{E} as in (2.17).

We need to complement the equations (5.11), (5.12) with the equations that determine the functions λ , μ (cf. (2.5), (2.6)). Notice that the functions ρ and p have the form:

$$\rho = \frac{\pi}{r^2} \tilde{E} B_1 \quad , \quad p = \frac{\pi}{r^2} \frac{v_1^2}{\tilde{E}} B_1 \quad , \quad r > R_+(t) \quad (5.13)$$

Then (2.5), (2.6) become:

$$e^{-2\lambda} (2r\lambda_r - 1) + 1 = 8\pi^2 \tilde{E} B_1, \quad e^{-2\lambda} (2r\mu_r + 1) - 1 = \frac{8\pi^2 v_1^2 B_1}{\tilde{E}} \quad (5.14)$$

We need to complement the system (5.11)-(5.14) with suitable boundary conditions. Imposing continuity for the velocities at $r = R_+(t)$ as well as for the fields λ , $\bar{\mu}$ and the densities we obtain:

$$v_1(\tau, r_+(\tau)) = e^\tau V_1(r_+(\tau) e^\tau) \quad , \quad B_1(\tau, r_+(\tau)) = e^{-\tau} b_1(r_+(\tau) e^\tau) \quad (5.15)$$

$$\lambda(t, r_+(\tau)) = \Lambda(r_+(\tau) e^\tau) \quad , \quad \bar{\mu}(t, r_+(\tau)) = \log(r_+(\tau)) + \hat{U}(r_+(\tau) e^\tau) \quad (5.16)$$

In order to require the weakest possible differentiability properties for the fields λ, μ , it is convenient to rewrite the equations (5.11), (5.12). Adding and subtracting $e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r \lambda$ to the left-hand side of (5.11) we obtain:

$$\partial_\tau v_1 + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r v_1 + Z(t, r, v_1) v_1 + e^{\bar{\mu}-\lambda} \left(\mu_r \tilde{E} - \frac{v_1^2}{\tilde{E}} \lambda_r \right) - e^{\bar{\mu}-\lambda} \frac{1}{r^3 \tilde{E}} = 0 \quad (5.17)$$

with $Z(t, r, v_1) = \left(\lambda_\tau + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r \lambda \right)$. Notice now that the term $\left(\mu_r \tilde{E} - \frac{v_1^2}{\tilde{E}} \lambda_r \right)$ can be rewritten, using (2.5), (2.6) and (5.13) in the form:

$$\mu_r \tilde{E} - \frac{v_1^2}{\tilde{E}} \lambda_r = \frac{(e^{2\lambda} - 1)}{2r \tilde{E}} \left(\frac{1}{r^2} + 2v_1^2 \right)$$

where we use the cancellation of the terms containing \tilde{B}_1 . Using this identity in (5.17) we obtain:

$$\partial_\tau v_1 + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r v_1 + Z(t, r, v_1) v_1 + \frac{(e^{2\lambda} - 1) e^{\bar{\mu}-\lambda}}{2r \tilde{E}} \left(\frac{1}{r^2} + 2v_1^2 \right) - e^{\bar{\mu}-\lambda} \frac{1}{r^3 \tilde{E}} = 0 \quad (5.18)$$

On the other hand adding and subtracting $e^{\bar{\mu}-\lambda} \mu_r \frac{v_1}{\tilde{E}}$ in the left-hand side of (5.12) we obtain, after some rearrangement of terms:

$$\partial_\tau B_1 + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r B_1 + \left(\frac{e^{\bar{\mu}-\lambda}}{\tilde{E}^3 r^2} (\partial_r v_1) + Z(t, r, v_1) + \frac{e^{\bar{\mu}-\lambda} (\mu_r - \lambda_r) v_1}{\tilde{E}} + \frac{e^{\bar{\mu}-\lambda} v_1}{r^3 \tilde{E}^3} \right) B_1 = 0 \quad (5.19)$$

A relevant property of (5.18) is that the terms containing derivatives of the fields $\lambda, \bar{\mu}$ appear in the form of the convective derivative $Z(t, r, v_1) = \left(\lambda_\tau + e^{\bar{\mu}-\lambda} \frac{v_1}{\tilde{E}} \partial_r \lambda \right)$. Then, we can remove this term from the equation by means of a change of variables, namely:

$$v_1(t, r) = \exp(-\lambda) \bar{w}(t, r) \quad , \quad B_1(t, r) = \exp(-\lambda) \bar{D}(t, r) \quad (5.20)$$

We remark that the change of variables (5.20) will play a role similar to (3.18) in Definition 13. Its goal is to eliminate terms like λ_t in the differential equations under consideration.

Then (5.18), (5.19) become:

$$\partial_\tau \bar{w} + e^{\bar{\mu}-2\lambda} \frac{\bar{w}}{\tilde{E}} \partial_r \bar{w} + \frac{e^{\bar{\mu}}}{r \tilde{E}} \left(\frac{(e^{2\lambda} - 1)}{2} \left(\frac{1}{r^2} + 2e^{-2\lambda} \bar{w}^2 \right) - \frac{1}{r^2} \right) = 0 \quad (5.21)$$

$$\partial_\tau \bar{D} + e^{\bar{\mu}-2\lambda} \frac{\bar{w}}{\tilde{E}} \partial_r \bar{D} + \frac{e^{\bar{\mu}-2\lambda}}{\tilde{E}} \left(\frac{\partial_r \bar{w}}{\tilde{E}^2 r^2} - \frac{\lambda_r \bar{w}}{\tilde{E}^2 r^2} + (\mu_r - \lambda_r) \bar{w} + \frac{w}{r^3 \tilde{E}^2} \right) \bar{D} = 0 \quad (5.22)$$

with:

$$\tilde{E} = \sqrt{\bar{w}^2 e^{-2\lambda} + \frac{1}{r^2}} \quad , \quad \tau = \log \left(\frac{1}{(-t)} \right) \quad (5.23)$$

The system (5.21), (5.22) must be solved with the following boundary conditions (cf. (5.15), (5.16), (5.20)):

$$\bar{w}(\tau, r_+(\tau)) = e^{\tau - \Lambda(r_+(\tau)e^\tau)} V_1(r_+(\tau)e^\tau) \quad , \quad \bar{D}(\tau, r_+(\tau)) = e^{-\tau - \Lambda(r_+(\tau)e^\tau)} b_1(r_+(\tau)e^\tau) \quad (5.24)$$

It is relevant to remark that the equation for \bar{D} contains a term involving a first derivative of v_1 , or more precisely $\frac{e^{\bar{\mu}-2\lambda}}{\tilde{E}_k^3 r^2} (\partial_r \bar{w})$. As a consequence we will need to consider function spaces which estimate one derivative more for \bar{D} than for \bar{w} .

A similar computation yields:

$$\left(\partial_\tau \bar{w}_k + e^{\bar{\mu}-2\lambda} \frac{\bar{w}_k}{\tilde{E}_k} \partial_r \bar{w}_k \right) + \frac{e^{\bar{\mu}}}{r \tilde{E}_k} \left(\frac{(e^{2\lambda} - 1)}{2} \left(\frac{1}{r^2} + 2e^{-2\lambda} \bar{w}_k^2 \right) - \frac{1}{r^2} \right) = 0 \quad (5.25)$$

$$\left(\partial_\tau \bar{D}_k + e^{\bar{\mu}-2\lambda} \frac{\bar{w}_k}{\tilde{E}_k} \partial_r \bar{D}_k \right) + \frac{e^{\bar{\mu}-2\lambda}}{\tilde{E}_k} \left(\frac{\partial_r \bar{w}_k}{\tilde{E}_k^2 r^2} - \frac{\lambda_r \bar{w}_k}{\tilde{E}_k^2 r^2} + (\mu_r - \lambda_r) \bar{w}_k + \frac{w_k}{r^3 \tilde{E}_k^2} \right) \bar{D}_k = 0 \quad (5.26)$$

for $r < R_+(t)$, $k = 1, 2$ with:

$$\tilde{E}_k = \sqrt{\bar{w}_k^2 e^{-2\lambda} + \frac{1}{r^2}} \quad , \quad \tau = \log \left(\frac{1}{(-t)} \right) \quad (5.27)$$

$$\frac{e^{-2\lambda} (2r\lambda_r - 1) + 1}{8\pi^2} = \left(\tilde{E}_1 B_1 + \tilde{E}_2 B_2 \right) \quad , \quad \frac{e^{-2\lambda} (2r\mu_r + 1) - 1}{8\pi^2} = \left(\frac{v_1^2 B_1}{\tilde{E}_1} + \frac{v_2^2 B_2}{\tilde{E}_2} \right) \quad (5.28)$$

for $r < R_+(t)$ and

$$v_k(t, r) = \exp(-\lambda) \bar{w}_k(t, r) \quad , \quad B_k(t, r) = \exp(-\lambda) \bar{D}_k(t, r) \quad (5.29)$$

if $k = 1, 2$, $r < R_+(t)$.

5.3 Characteristic curves for (5.21)-(5.23).

Our next goal is to solve the equations (5.21)-(5.23). We also want to define a concept of solution of (5.21)-(5.24) using the weakest possible regularity. To this end, we will integrate these equations using characteristics. We now formulate the characteristic equations. The solvability of these equations will be proved later.

Let us denote as $(r(\tau; \bar{\tau}), w(\tau; \bar{\tau}), D(\tau; \bar{\tau}))$ the characteristic curves associated to the system (5.21), (5.22) defined for $\tau \leq \bar{\tau}$. We will assume that the curve $\{r = r(\tau; \bar{\tau})\}$ reaches the boundary of the domain $\mathcal{D}(T) = \{(\tau, r) : r > r_+(\tau); \tau \geq T\}$ for $\tau = \bar{\tau}$. Suppose that the curves $\{r = r(\tau; \bar{\tau}) : T \leq \tau \leq \bar{\tau}\}$ cover the whole domain $\mathcal{D}(T)$. Then the derivative $\partial_r w$ in (5.22) evaluated at $(r(\tau; \bar{\tau}), \tau)$ can be computed, using the Implicit Function Theorem, by means of $(\frac{\partial w}{\partial \tau}(\tau, \bar{\tau})) / (\frac{\partial r}{\partial \tau}(\tau, \bar{\tau}))$.

We can then write the characteristic equations associated to the equations (5.21)-(5.23) as:

$$\frac{\partial r(\tau; \bar{\tau})}{\partial \tau} = \frac{e^{\bar{\mu}(r(\tau; \bar{\tau}), \tau) - 2\lambda(r(\tau; \bar{\tau}), \tau)} w(\tau; \bar{\tau}) r(\tau; \bar{\tau})}{\Xi(\tau; \bar{\tau})}, \quad r(\bar{\tau}; \bar{\tau}) = r_+(\bar{\tau}) \quad (5.30)$$

$$\begin{aligned} \frac{\partial w(\tau; \bar{\tau})}{\partial \tau} = & -\frac{e^{\bar{\mu}(r(\tau; \bar{\tau}), \tau)} (e^{2\lambda(r(\tau; \bar{\tau}))} - 1)}{2\Xi(\tau; \bar{\tau})} \left[\frac{1}{(r(\tau; \bar{\tau}))^2} + 2e^{-2\lambda(r(\tau; \bar{\tau}), \tau)} (w(\tau; \bar{\tau}))^2 \right] \\ & + \frac{e^{\bar{\mu}(r(\tau; \bar{\tau}), \tau)}}{(r(\tau; \bar{\tau}))^2 \Xi(\tau; \bar{\tau})} \end{aligned} \quad (5.31)$$

$$w(\bar{\tau}; \bar{\tau}) = \exp(\bar{\tau} - \Lambda(r_+(\bar{\tau}) e^{\bar{\tau}})) V_1(r_+(\bar{\tau}) e^{\bar{\tau}}) \quad (5.32)$$

$$\begin{aligned} & \frac{\partial D(\tau; \bar{\tau})}{\partial \tau} \quad (5.33) \\ & = - \left(\frac{e^{\bar{\mu}(r(\tau; \bar{\tau}), \tau) - 2\lambda(r(\tau; \bar{\tau}), \tau)} \cdot \left[\frac{\left(\frac{\partial w}{\partial \tau}(\tau, \bar{\tau}) \right)}{\left(\frac{\partial r}{\partial \tau}(\tau, \bar{\tau}) \right)} - (\partial_r \lambda)(r(\tau; \bar{\tau}), \tau) w(\tau; \bar{\tau}) \right] r(\tau; \bar{\tau})}{(\Xi(\tau; \bar{\tau}))^3} \right. \\ & \quad + \frac{e^{\bar{\mu}(r(\tau; \bar{\tau}), \tau) - 2\lambda(r(\tau; \bar{\tau}), \tau)} [\bar{\mu}_r(r(\tau; \bar{\tau}), \tau) - \lambda_r(r(\tau; \bar{\tau}), \tau)] w(\tau; \bar{\tau}) r(\tau; \bar{\tau})}{\Xi(\tau; \bar{\tau})} \\ & \quad \left. + \frac{e^{\bar{\mu}(r(\tau; \bar{\tau}), \tau) - 2\lambda(r(\tau; \bar{\tau}), \tau)} w(\tau; \bar{\tau})}{(\Xi(\tau; \bar{\tau}))^3} \right) D(\tau; \bar{\tau}) \\ & D(\bar{\tau}; \bar{\tau}) = \exp(-\bar{\tau} - \Lambda(r_+(\bar{\tau}) e^{\bar{\tau}})) b_1(r_+(\bar{\tau}) e^{\bar{\tau}}) \end{aligned}$$

where the boundary values have been chosen using (5.24) and where:

$$\Xi(\tau; \bar{\tau}) = \sqrt{1 + e^{-2\lambda(r(\tau; \bar{\tau}), \tau)} (w(\tau; \bar{\tau}))^2 (r(\tau; \bar{\tau}))^2} \quad (5.34)$$

We are interested in obtaining solutions of (5.30)-(5.33) in the domain:

$$\mathcal{U}(T) = \{(\tau, \bar{\tau}) \in (T, \infty) \times (T, \infty) : \bar{\tau} \geq \tau\} \quad (5.35)$$

where from now on $T = \log\left(\frac{1}{(-t_0)}\right)$.

In order to solve the system (5.30)-(5.33) we need to define suitable function spaces. As a preliminary step we study the asymptotic behavior or the solutions of a system of equations which will describe the asymptotics of the solutions of (5.30)-(5.32) for large values of τ and $\bar{\tau}$.

5.4 Formal asymptotic behaviour of the characteristic curves (5.30)-(5.32).

In this Section we derive by means of formal computations the asymptotics of the solutions of (5.11), (5.12), (5.14), (5.15), (5.16) that we construct in this

paper. These computations, although formal, will be useful to get some intuitive understanding of the form of the solutions and also to give some justification of the function spaces that we will use to construct the solutions. We also study some auxiliary functions which will be used in the following to define suitable function spaces.

Notice that we can expect $\bar{D}(t, r)$ to approach to zero as $t \rightarrow 0$, due to (5.24). Then, using (5.20) as well as the fact that the fields λ, μ are given as in Lemma 2 and the asymptotics (5.7), we would obtain the following approximations for $\bar{\tau} \geq \tau \geq T$ if τ_0 is large enough:

$$r_+(\bar{\tau}) = R_{\max} \quad , \quad R_0(\tau, r) = \frac{2R_{\max}}{3}$$

$$\lambda(\tau, r) = \lambda_0(r) = \frac{1}{2} \log(r) - \frac{1}{2} \log\left(r - \frac{2R_{\max}}{3}\right) \quad (5.36)$$

$$\bar{\mu}(\tau, r) = \bar{\mu}_0(r) = \log\left(\frac{R_{\max} \sqrt{3(1-y_0^2)}}{y_0}\right) + \frac{1}{2} \log\left(1 - \frac{2R_{\max}}{3r}\right) \quad (5.37)$$

With these approximations, (5.30)-(5.32) become:

$$\frac{\partial r(\tau; \bar{\tau})}{\partial \tau} = \frac{e^{\bar{\mu}_0(r(\tau; \bar{\tau})) - 2\lambda_0(r(\tau; \bar{\tau}))} w(\tau; \bar{\tau}) r(\tau; \bar{\tau})}{\Xi(\tau; \bar{\tau})} \quad , \quad r(\bar{\tau}; \bar{\tau}) = R_{\max} \quad (5.38)$$

$$\begin{aligned} \frac{\partial w(\tau; \bar{\tau})}{\partial \tau} = & -\frac{e^{\bar{\mu}_0(r(\tau; \bar{\tau}))} (e^{2\lambda_0(r(\tau; \bar{\tau}))} - 1)}{2\Xi(\tau; \bar{\tau})} \left[\frac{1}{(r(\tau; \bar{\tau}))^2} + 2e^{-2\lambda_0(r(\tau; \bar{\tau}), \tau)} (w(\tau; \bar{\tau}))^2 \right] \\ & + \frac{e^{\bar{\mu}_0(r(\tau; \bar{\tau}))}}{(r(\tau; \bar{\tau}))^2 \Xi(\tau; \bar{\tau})} \end{aligned} \quad (5.39)$$

$$w(\bar{\tau}; \bar{\tau}) = -\frac{6y_0 \sqrt{(1-y_0^2)}}{(1-4y_0^4) R_{\max}}$$

with $\Xi(\tau; \bar{\tau})$ as in (5.34). The solution of (5.38), (5.39) can be obtained in the form:

$$r(\tau; \bar{\tau}) = \mathcal{R}(\tau - \bar{\tau}) \quad , \quad w(\tau; \bar{\tau}) = \mathcal{W}(\tau - \bar{\tau})$$

where \mathcal{R} and \mathcal{W} solve:

$$\frac{\partial \mathcal{R}(\tau)}{\partial \tau} = \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau)) - 2\lambda_0(\mathcal{R}(\tau))} \mathcal{W}(\tau) \mathcal{R}(\tau)}{\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau))^2 (\mathcal{W}(\tau))^2}}, \quad \mathcal{R}(0) = R_{\max} \quad (5.40)$$

$$\begin{aligned} \frac{\partial \mathcal{W}(\tau)}{\partial \tau} = & -\frac{e^{\bar{\mu}_0(\mathcal{R}(\tau))} (e^{2\lambda_0(\mathcal{R}(\tau))} - 1)}{2\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau))^2 (\mathcal{W}(\tau))^2}} \left[\frac{1}{(\mathcal{R}(\tau))^2} + 2e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{W}(\tau))^2 \right] \\ & + \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau))}}{(\mathcal{R}(\tau))^2 \sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau))^2 (\mathcal{W}(\tau))^2}} \end{aligned} \quad (5.41)$$

$$\mathcal{W}(0) = -\frac{6y_0\sqrt{(1-y_0^2)}}{(1-4y_0^2)R_{\max}} \quad (5.42)$$

The functions \mathcal{R} and \mathcal{W} will play an important role in the following in order to describe the function spaces used in the solution of (5.30)-(5.33). In the next proposition we describe their asymptotic properties.

Proposition 24 *Suppose that $(\mathcal{R}, \mathcal{W})$ solve (5.40)-(5.42) for $\tau \leq 0$. Then:*

$$\mathcal{W}(\tau) \rightarrow -\sqrt{\frac{4y_0^2(1-y_0^2)}{(1-4y_0^2)^2 R_{\max}^2} + \frac{1}{R_{\max}}} \quad \text{as } \tau \rightarrow -\infty \quad (5.43)$$

$$\mathcal{R}(\tau) \sim -\frac{R_{\max}\sqrt{3(1-y_0^2)}}{y_0}\tau \quad \text{as } \tau \rightarrow -\infty \quad (5.44)$$

We have also:

$$|\mathcal{R}'(\tau)| \leq C(y_0, R_{\max}) \quad \text{for } \tau \leq 0 \quad (5.45)$$

$$|\mathcal{W}(\tau)| \leq C(y_0, R_{\max}) \quad \text{for } \tau \leq 0 \quad (5.46)$$

for some constant $C(y_0, R_{\max})$ depending only on y_0, R_{\max} . Moreover, there exists $\Gamma(y_0, R_{\max}) > 0$, depending only on y_0, R_{\max} such that:

$$\mathcal{W}(\tau) \leq -\Gamma(y_0, R_{\max}) \quad \text{for } \tau \leq 0 \quad (5.47)$$

Proof. The system of equations (5.40)-(5.42) can be solved explicitly. Indeed, this system can be reformulated as a Hamiltonian system. To this end we define a new variable by means of:

$$\mathcal{Z}(\tau) = \mathcal{W}(\tau) \frac{(\mathcal{R}(\tau) - \frac{2R_{\max}}{3})}{\mathcal{R}(\tau)} \quad (5.48)$$

Using (5.40), (5.41) and (5.48) we then obtain, after some computations:

$$\frac{\partial \mathcal{Z}(\tau)}{\partial \tau} = \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau))}}{\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau))^2 (\mathcal{W}(\tau))^2}} \left[-\frac{R_{\max}}{(\mathcal{R}(\tau))^2} + \frac{1}{\mathcal{R}(\tau)} \right] \quad (5.49)$$

We rewrite (5.40) as:

$$\frac{\partial \mathcal{R}(\tau)}{\partial \tau} = \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau))} \mathcal{Z}(\tau) \mathcal{R}(\tau)}{\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau))^2 (\mathcal{W}(\tau))^2}} \quad (5.50)$$

Combining (5.50) and (5.49) we can obtain a conserved quantity along characteristics, namely:

$$\frac{(\mathcal{Z}(\tau))^2}{2} - \frac{R_{\max}}{2(\mathcal{R}(\tau))^2} + \frac{1}{\mathcal{R}(\tau)}$$

Using the value of $\mathcal{R}(0)$ in (5.40), $\mathcal{W}(0)$ in (5.42) and (5.48) we obtain:

$$\frac{(\mathcal{Z}(\tau))^2}{2} - \frac{R_{\max}}{2(\mathcal{R}(\tau))^2} + \frac{1}{\mathcal{R}(\tau)} = \frac{2y_0^2(1-y_0^2)}{(1-4y_0^2)^2 R_{\max}^2} + \frac{1}{2R_{\max}} \quad (5.51)$$

Notice that, since $\mathcal{Z}(0) = -\frac{2y_0\sqrt{(1-y_0^2)}}{(1-4y_0^2)R_{\max}} < 0$ it follows from (5.49) that $\mathcal{Z}(\tau) < 0$ for $\tau < \bar{\tau}$. On the other hand, (5.50) implies that $\mathcal{R}(\tau)$ is decreasing and then $\mathcal{R}(\tau) > R_{\max}$ for $\tau < \bar{\tau}$. Moreover, (5.50) implies also that $\frac{\partial \mathcal{R}(\tau)}{\partial \tau}$ remains of order one for $\tau < \bar{\tau}$ and then $\mathcal{R}(\tau) \rightarrow \infty$ as $(\bar{\tau} - \tau) \rightarrow \infty$. Therefore:

$$\mathcal{Z}(\tau) \rightarrow -\sqrt{\frac{4y_0^2(1-y_0^2)}{(1-4y_0^2)^2 R_{\max}^2} + \frac{1}{R_{\max}}} \quad \text{as } \tau \rightarrow -\infty \quad (5.52)$$

Then (5.48) implies (5.43). It then follows from (5.50), using also that $\lambda_0(\mathcal{R}(\tau)) \rightarrow 0$ as $\tau \rightarrow -\infty$ and $\bar{\mu}_0(\mathcal{R}(\tau)) \rightarrow \log\left(\frac{R_{\max}\sqrt{3(1-y_0^2)}}{y_0}\right)$ as $\tau \rightarrow -\infty$ that:

$$\frac{\partial \mathcal{R}(\tau)}{\partial \tau} \rightarrow -\frac{R_{\max}\sqrt{3(1-y_0^2)}}{y_0} \quad \text{as } \tau \rightarrow -\infty \quad (5.53)$$

whence (5.44) follows. Moreover, since $\frac{\partial \mathcal{R}(\tau)}{\partial \tau}$ is bounded in bounded regions, this implies also (5.45). Using also that \mathcal{W} is bounded as well as (5.43) we obtain (5.46). To prove (5.47) we use the fact that since $\mathcal{Z}(0) = -\frac{2y_0\sqrt{(1-y_0^2)}}{(1-4y_0^2)R_{\max}} < 0$, and (5.52) as well as the fact that $\mathcal{Z}(\tau) < 0$ for $\tau < \bar{\tau}$ imply that $\mathcal{Z}(\tau) \leq -\Gamma_*(y_0, R_{\max}) < 0$ for $\tau < \bar{\tau}$. Using then (5.48) as well as the fact that $\frac{(\mathcal{R}(\tau) - \frac{2R_{\max}}{3})}{\mathcal{R}(\tau)} < 1$ for $\mathcal{R}(\tau) \geq R_{\max}$, we obtain (5.47). This concludes the proof. ■

5.5 Function spaces.

We define a function space $\mathcal{X}_{L,T}$ as follows. Suppose that

$$r, w, D \in W_{\text{loc}}^{1,\infty}(\mathcal{U}(T)) \quad (5.54)$$

where $\mathcal{U}(T)$ is as in (5.35). Let us assume also that the functions r, w, D satisfy:

$$0 \leq D(\tau; \bar{\tau}) \exp(2\bar{\tau}) \leq L \quad , \quad a.e. \quad (\tau, \bar{\tau}) \in \mathcal{U}(T) \quad (5.55)$$

$$\left[\frac{|\rho^*(\tau; \bar{\tau})|}{1 + (\bar{\tau} - \tau)} + |z(\tau; \bar{\tau})| \right] \leq \frac{1}{L} \quad , \quad a.e. \quad (\tau, \bar{\tau}) \in \mathcal{U}(T) \quad (5.56)$$

where ρ^*, z are defined by means of:

$$\rho^*(\tau; \bar{\tau}) = (r(\tau; \bar{\tau}) - \mathcal{R}(\tau - \bar{\tau})) \quad , \quad z(\tau; \bar{\tau}) = (w(\tau; \bar{\tau}) - \mathcal{W}(\tau - \bar{\tau})) \quad , \quad \bar{\tau} \geq \tau \geq T \quad (5.57)$$

On the other hand, since $r, w \in W_{\text{loc}}^{1,\infty}(\mathcal{U}(T))$ they are differentiable *a.e.* in $\mathcal{U}(T)$. We will assume also that:

$$\max \left\{ \left| \frac{\partial \rho^*(\tau; \bar{\tau})}{\partial \bar{\tau}} \right|, \left| \frac{\partial z(\tau; \bar{\tau})}{\partial \bar{\tau}} \right| \right\} \leq \frac{1}{L} \quad , \quad a.e. \quad \text{in } \mathcal{U}(T) \quad (5.58)$$

$$\frac{\partial r}{\partial \tau}(\tau; \bar{\tau}) \leq -\frac{1}{L} \quad , \quad a.e. \quad \text{in } \mathcal{U}(T) \quad (5.59)$$

We will assume also the following estimates for the derivatives $\frac{\partial r}{\partial \tau}, \frac{\partial w}{\partial \tau}$:

$$\max \left\{ \left| \frac{\partial r}{\partial \tau} \right|, \left| \frac{\partial w}{\partial \tau} \right| \right\} \leq \sqrt{L} \quad , \quad a.e. \quad \text{in } \mathcal{U}(T) \quad (5.60)$$

We will assume also that for the functions in the space $\mathcal{X}_{L,T}$ we have:

$$r(\bar{\tau}; \bar{\tau}) = r_+(\bar{\tau}) \quad , \quad w(\bar{\tau}; \bar{\tau}) = \exp(\bar{\tau} - \Lambda(r_+(\bar{\tau})e^{\bar{\tau}})) V_1(r_+(\bar{\tau})e^{\bar{\tau}}) \quad , \quad T \leq \bar{\tau} < \infty \quad (5.61)$$

It is relevant to remark that the right-hand sides of these equations are smooth functions. Therefore, the Lipschitz property implied by (5.58), (5.60) is satisfied on the line $\{(\tau, \bar{\tau}) \in \mathcal{U}(T) : \tau = \bar{\tau}\}$.

Definition 25 We will denote the space of functions satisfying (5.54), (5.56), (5.58), (5.59), (5.60) and (5.61) by $\mathcal{X}_{L,T}$.

Remark 26 We will use in the following that, since y_0 can be assumed to be small, the exponent $\gamma(y_0) = \left(1 - \frac{4(1-y_0^2)}{(1-4y_0^2)}\right)$ is larger than 2 in absolute value.

We remark also that (5.57), (5.58), (5.60) imply that the functions r, w are Lipschitz continuous. Then they are differentiable *a.e.*

Given any $(r, w, D) \in \mathcal{X}_{L,T}$ we can construct some auxiliary functions $\bar{w}(\tau, r), \bar{D}(\tau, r)$ defined for $r \geq r_+(\tau), T \leq \tau < \infty$. This is proved in the following Lemma.

Lemma 27 *There exist $L_0 > 0$ and $T_0 > 0$ sufficiently large such that, for any function $(r, w, D) \in \mathcal{X}_{L,T}$ with $L > L_0$ and $T > T_0$ there exist functions $\bar{D} \in L^\infty$ and $\bar{w} \in W^{1,\infty}$ defined for $r \geq r_+(\tau)$, $\tau \geq T$ such that:*

$$\bar{D}(\tau, r(\tau; \bar{\tau})) = D(\tau; \bar{\tau}) \quad , \quad r = r(\tau; \bar{\tau}) \quad , \quad \bar{\tau} \geq \tau \geq T \quad , \quad r \geq r_+(\tau) \quad (5.62)$$

$$\bar{w}(\tau, r(\tau; \bar{\tau}), \tau) = w(\tau; \bar{\tau}) \quad , \quad r = r(\tau; \bar{\tau}) \quad , \quad \bar{\tau} \geq \tau \geq T \quad , \quad r \geq r_+(\tau) \quad (5.63)$$

There exist positive constants C_0 , a depending only on y_0 but independent of L such that:

$$0 \leq \bar{D}(\tau, r) \leq C_0 L e^{-2\tau} e^{-ar} \quad , \quad |\bar{w}(\tau, r)| \leq C_0 \quad , \quad r \geq r_+(\tau) \quad (5.64)$$

Moreover, we have also the inequality:

$$r(\tau; \bar{\tau}) \leq r_+(\bar{\tau}) + 2C_0(\bar{\tau} - \tau) \quad , \quad \bar{\tau} \geq \tau \geq T \quad (5.65)$$

Proof. We will assume in all the following that $L \geq 1$. Given $r > r_+(\tau)$ we define $\bar{\tau}(\tau, r)$ by means of the formula:

$$r = r(\tau; \bar{\tau}(\tau, r)) = \mathcal{R}(\tau - \bar{\tau}(\tau, r)) + [r(\tau; \bar{\tau}(\tau, r)) - \mathcal{R}(\tau - \bar{\tau}(\tau, r))]$$

In order to prove that the function $\bar{\tau}(\tau, r)$ is well defined, we remark that, due to the definition of the space $\mathcal{X}_{L,T}$ we have, for $T \leq \tau \leq \bar{\tau} < \infty$ and L sufficiently large, the inequality $\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \geq \theta > 0$ where the number θ is independent of L . Indeed, since $\mathcal{R}'(s) \leq -\delta < 0$ for $s \leq 0$ it follows that, choosing L_0 sufficiently large we obtain:

$$\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \geq \frac{\delta}{2} > 0 \quad (5.66)$$

where we use again the properties of the functions $r(\cdot, \cdot) \in \mathcal{X}_{L,T}$.

Therefore using that $r_+(\bar{\tau}) = r(\bar{\tau}; \bar{\tau})$, assuming that $\tau \geq T$ is fixed, it follows that the image of the mapping defined by means of $\bar{\tau} \rightarrow r(\tau; \bar{\tau})$ covers the whole range of values $[r_+(\tau), \infty)$. Therefore there exists a unique value $\bar{\tau} = \bar{\tau}(\tau, r)$ such that:

$$r = r(\tau; \bar{\tau}(\tau, r)) \quad (5.67)$$

We then define functions \bar{D} and \bar{w} by means of:

$$\bar{D}(\tau, r) = D(\tau; \bar{\tau}(\tau, r)) \quad , \quad \bar{w}(\tau, \bar{\tau}(\tau, r)) = w(\tau; \bar{\tau}) \quad (5.68)$$

It remains to check the regularity properties of the function \bar{v}_1 . To this end we need to prove regularity for $\bar{\tau}(\tau, r)$. Notice that due to the definition of $\mathcal{X}_{L,T}$ we have estimates of the form $0 < C_1 \leq \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \leq C_2$ assuming that L_0 is large and $T \leq \tau$. This implies that the function $\bar{\tau}(\tau, r)$ is uniformly Lipschitz in the variable r for $\tau \geq T$. Suppose that:

$$r(\tau; \bar{\tau}_1) = r_1 \quad , \quad r(\tau; \bar{\tau}_2) = r_2$$

Suppose that $r_2 \geq r_1$. Then:

$$r_2 - r_1 = r(\tau; \bar{\tau}_2) - r(\tau; \bar{\tau}_1) \geq C_1 (\bar{\tau}_2 - \bar{\tau}_1)$$

whence $(\bar{\tau}_2 - \bar{\tau}_1) \leq \frac{1}{C_1} (r_2 - r_1)$. Therefore, since the choice of the largest r_k is arbitrary:

$$|\bar{\tau}(\tau, r_2) - \bar{\tau}(\tau, r_1)| \leq C |r_2 - r_1|$$

On the other hand, the function v_1 is Lipschitz by assumption. Using the strict monotonicity of the function $\bar{\tau}(\tau, \cdot)$ and the Lipschitz property of both v_1 and $\bar{\tau}(\cdot, \tau)$ we can prove, arguing as in the proof of the chain rule, that $\bar{v}_1(\tau, \cdot)$ is Lipschitz uniformly for $\tau \geq T$.

We also need to prove the Lipschitz property in the variable τ . Arguing again as in the proof of the chain rule, we can see that the problem reduces to prove that the function $\bar{\tau}(\tau, r)$ is Lipschitz with respect to τ . To this end we begin with the formula:

$$r = r(\tau_1; \bar{\tau}(\tau_1, r)) \quad , \quad r = r(\tau_2; \bar{\tau}(\tau_2, r))$$

Then:

$$\begin{aligned} r(\tau_1; \bar{\tau}(\tau_1, r)) &= r(\tau_2; \bar{\tau}(\tau_2, r)) \\ r(\tau_1; \bar{\tau}(\tau_1, r)) - r(\tau_1; \bar{\tau}(\tau_2, r)) &= r(\tau_2; \bar{\tau}(\tau_2, r)) - r(\tau_1; \bar{\tau}(\tau_2, r)) \end{aligned}$$

Using the Lipschitz estimate of $r(\tau, \bar{\tau})$ in τ we can estimate the right-hand side. On the other hand, suppose that $\bar{\tau}(\tau_1, r) \geq \bar{\tau}(\tau_2, r)$. The estimates for r imply:

$$C_1 (\bar{\tau}(\tau_1, r) - \bar{\tau}(\tau_2, r)) \leq C |\tau_2 - \tau_1|$$

Then:

$$|\bar{\tau}(\tau_1, r) - \bar{\tau}(\tau_2, r)| \leq C |\tau_2 - \tau_1| \quad (5.69)$$

Therefore $\bar{\tau}$ is globally Lipschitz and then $\bar{D}, \bar{w} \in W^{1, \infty}$.

In order to prove (5.64) we notice that (5.45) and (5.58) imply:

$$|r(\tau; \bar{\tau}) - r(\bar{\tau}; \bar{\tau})| \leq 2C(y_0)(\bar{\tau} - \tau) \quad , \quad \bar{\tau} \geq \tau$$

since $L \geq 1$. Using (5.61) as well as (5.67) we derive (5.65).

Due to (5.55), (5.68) we have $0 \leq \bar{D}(\tau, r) \leq L \exp(-2\bar{\tau}(\tau, r))$. Then, using the boundedness of $r_+(\bar{\tau})$ as well as the fact that $L \geq 1$ we obtain $\bar{D}(\tau, r) \leq CLe^{-2\tau}e^{-ar}$ for some $a > 0$, depending on y_0 but independent of L . This gives the first inequality in (5.64). The second inequality follows from (5.46) and (5.56). ■

Lemma 27 suggests introducing the following function spaces:

Definition 28 We define the space $\mathcal{Y}_{L, T, a}$ as the space of functions $\bar{D} \in L^\infty(\{(\tau, r) : r \geq r_+(\tau), \tau \geq T\})$ satisfying the inequality:

$$0 \leq \bar{D}(\tau, r) \leq L^{\frac{3}{2}} e^{-2\tau} e^{-ar} \quad (5.70)$$

with $a > 0$.

Remark 29 In the remainder of this paper it will be assumed that, given L, T the constant a used in the definition of the spaces $\mathcal{Y}_{L, T, a}$ is chosen as indicated in Lemma 27 whenever these spaces are referred to.

5.6 Solutions of (5.21)-(5.24) in the sense of characteristics.

We now define a suitable concept of solution of (5.21)-(5.24).

Definition 30 Suppose that $L > L_0$, $T > T_0$ with L_0, T_0 as in Lemma 27. We will say that $((r, w), D) \in \mathcal{X}_{L,T}$ is a solution of (5.21)-(5.24) in the sense of characteristics if it satisfies (5.30)-(5.33) a.e. $(\tau, \bar{\tau}) \in \mathcal{U}(T)$.

Our next goal is to prove that a solution of (5.21)-(5.24) in the sense of characteristics allows to obtain a solution of (2.5)-(2.8), (2.16), (2.17) in the sense of Definition 13. As a first step we derive a formula relating $D(\tau; \bar{\tau})$ and $\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}}$. This formula provides some geometric interpretation for the function D which is related with the stretching of the characteristic curves. It will be used repeatedly in the following.

5.7 A representation formula for $D(\tau; \bar{\tau})$ in terms of $\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}}$.

5.7.1 Heuristics.

We first need to obtain a formula for $D(\tau, \bar{\tau})$ (or equivalently $D(t, \bar{t})$ with some abuse of notation). The equation of $D(\tau, \bar{\tau})$ can be integrated explicitly for given fields λ, μ if the corresponding functions $r(\tau, \bar{\tau}), w(\tau, \bar{\tau})$ are known. We first obtain formally a derivation of the desired formula. To this end, we first write the equation for ζ in divergence form. Using (2.16) we can compute $\partial_t(e^\lambda \zeta)$. After some simple, but tedious algebraic computations we obtain:

$$\begin{aligned} & \partial_t(e^\lambda \zeta) + \partial_r\left(e^\mu \frac{v}{\tilde{E}} \zeta\right) - \partial_r\left(\frac{e^\mu}{\tilde{E}}\right) v \zeta - \partial_v\left[e^\lambda \left(\lambda_t v + e^{\mu-\lambda} \mu_r \tilde{E} - e^{\mu-\lambda} \frac{1}{r^3 \tilde{E}}\right) \zeta\right] \\ & + \zeta \left(e^\mu \mu_r \partial_v \tilde{E} - \frac{e^\mu}{r^3} \partial_v \left(\frac{1}{\tilde{E}}\right)\right) \\ & = 0 \end{aligned} \tag{5.71}$$

Using the definition of \tilde{E} in (2.17) we obtain:

$$-\partial_r\left(\frac{e^\mu}{\tilde{E}}\right) + \left(e^\mu \mu_r \partial_v \tilde{E} - \frac{e^\mu}{r^3} \partial_v \left(\frac{1}{\tilde{E}}\right)\right) = 0 \tag{5.72}$$

Combining (5.71), (5.72) we obtain the following equation which has a suitable divergence form structure:

$$\partial_t(e^\lambda \zeta) + \partial_r\left(e^\mu \frac{v}{\tilde{E}} \zeta\right) - \partial_v\left[e^\lambda \left(\lambda_t v + e^{\mu-\lambda} \mu_r \tilde{E} - e^{\mu-\lambda} \frac{1}{r^3 \tilde{E}}\right) \zeta\right] = 0 \tag{5.73}$$

Notice that this formula is valid for any pair of functions λ, μ even if they are not related with the function \bar{D} by means of (5.14), (5.20).

We can apply (5.73) now formally to a measure $\zeta(t, r, v)$ with the form (5.10). Our goal now is to obtain a constant quantity along characteristics. Using (5.10) and the second identity in (5.20) we obtain:

$$\int_{r(t, \bar{t})}^{r(t, \bar{t} + \Delta \bar{t})} \int_0^\infty e^\lambda \zeta(t, r, v) dr dv = \int_{r(t, \bar{t})}^{r(t, \bar{t} + \Delta \bar{t})} \bar{D}(t, r) dr \quad (5.74)$$

where $\Delta \bar{t}$ is arbitrary. Using now (5.73) and integrating in the variable v we arrive at:

$$\partial_t \left(\int_0^\infty e^\lambda \zeta dv \right) + \partial_r \left(\int_0^\infty e^\mu \frac{v}{\tilde{E}} \zeta dv \right) = 0$$

which combined with (5.10) yields:

$$\partial_t (\bar{D}(t, r)) + \partial_r \left(e^{\mu - \lambda} \frac{v_1(t, r)}{\tilde{E}} \bar{D}(t, r) \right) = 0$$

We can use this formula to differentiate $\int_{r(t, \bar{t})}^{r(t, \bar{t} + \Delta \bar{t})} \bar{D}(t, r) dr$. Using that:

$$\frac{\partial r}{\partial t} = e^{\mu - \lambda} \frac{v_1(t, r)}{\tilde{E}}$$

we obtain:

$$\begin{aligned} \partial_t \left(\int_{r(t, \bar{t})}^{r(t, \bar{t} + \Delta \bar{t})} \bar{D}(t, r) dr \right) &= \bar{D}(t, r(t, \bar{t} + \Delta \bar{t})) \frac{\partial r(t, \bar{t} + \Delta \bar{t})}{\partial t} - \bar{D}(t, r(t, \bar{t})) \frac{\partial r(t, \bar{t})}{\partial t} \\ &\quad - \int_{r(t, \bar{t})}^{r(t, \bar{t} + \Delta \bar{t})} \partial_r \left(e^{\mu - \lambda} \frac{v_1(t, r)}{\tilde{E}} \bar{D}(t, r) \right) \end{aligned}$$

and, using the formula for $\frac{\partial r}{\partial t}$ it then follows that:

$$\partial_t \left(\int_{r(t, \bar{t})}^{r(t, \bar{t} + \Delta \bar{t})} \bar{D}(t, r) dr \right) = 0$$

This formula yields in an integral form the desired conservation law. In order to derive a pointwise conserved quantity we use the change of variables $r = r(t, \bar{t})$, $dr = \frac{\partial r(t, \bar{t})}{\partial \bar{t}} d\bar{t}$. Then:

$$\partial_t \left(\int_{\bar{t}}^{\bar{t} + \Delta \bar{t}} \bar{D}(t, r(t, \bar{t})) \frac{\partial r(t, \bar{t})}{\partial \bar{t}} d\bar{t} \right) = 0$$

and since $\Delta \bar{t}$ is arbitrary we deduce that:

$$\bar{D}(t, r(t, \bar{t})) \frac{\partial r(t, \bar{t})}{\partial \bar{t}} = D(t, \bar{t}) \frac{\partial r(t, \bar{t})}{\partial \bar{t}} = D(\bar{t}, \bar{t}) \frac{\partial r(\bar{t}, \bar{t})}{\partial \bar{t}} \quad (5.75)$$

This is the desired conservation law which we will derive now in a more precise and rigorous way.

5.7.2 Rigorous proof of the representation formula for $D(\tau; \bar{\tau})$.

Our goal is to prove (5.75) rigorously. The precise result is the following.

Lemma 31 *Suppose that $\lambda, \mu \in W^{1,\infty}(\{r \geq r_+(\tau), \tau \geq T\})$ and that $(r, w, D) \in \mathcal{X}_{L,T}$ solve (5.30)-(5.33). Then, there exists $f(\bar{\tau})$ such that:*

$$\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) = f(\bar{\tau}) \quad , \quad a.e. \quad (\tau; \bar{\tau}) \in \mathcal{U}(T) \quad (5.76)$$

Proof. If the functions $\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}}$, $D(\tau; \bar{\tau})$ were smooth, the result would follow from the cancellation of the derivative $\frac{d}{d\tau} \left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right]$. Given that our regularity assumptions do not guarantee the existence of this derivative in a classical sense, we need to compute this derivative in weak form. We will assume first that all the required derivatives exist and describe later how to adapt the argument to a weak formulation. We compute first $\frac{d}{d\tau} \left(\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \right)$.

Differentiating (5.30) and rearranging some terms we obtain:

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \right) &= \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} w(\tau; \bar{\tau}) r(\tau; \bar{\tau})}{\Xi(\tau; \bar{\tau})} [\bar{\mu}_r(\tau, r(\tau; \bar{\tau})) - 2\lambda_r(\tau, r(\tau; \bar{\tau}))] \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \\ &+ \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} r(\tau; \bar{\tau})}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial w(\tau; \bar{\tau})}{\partial \bar{\tau}} \\ &+ \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} w(\tau; \bar{\tau})}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \\ &+ \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 4\lambda(\tau, r(\tau; \bar{\tau}))} (w(\tau; \bar{\tau}))^3 (r(\tau; \bar{\tau}))^3 \lambda_r(\tau, r(\tau; \bar{\tau}))}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \end{aligned}$$

with $\Xi(\tau; \bar{\tau})$ as in (5.34).

Combining this equation with (5.33) we obtain:

$$\begin{aligned} &\frac{\partial}{\partial \tau} \left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right] \\ &= - \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} w(\tau; \bar{\tau}) r(\tau; \bar{\tau}) \lambda_r(\tau, r(\tau; \bar{\tau}))}{\Xi(\tau; \bar{\tau})} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \\ &+ \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} w(\tau; \bar{\tau}) r(\tau; \bar{\tau}) \lambda_r(\tau, r(\tau; \bar{\tau}))}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \\ &+ \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} r(\tau; \bar{\tau})}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial w(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \\ &- \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} \cdot (\partial_r w)(\tau, r(\tau; \bar{\tau})) r(\tau; \bar{\tau})}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \quad (5.77) \\ &+ \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 4\lambda(\tau, r(\tau; \bar{\tau}))} (w(\tau; \bar{\tau}))^3 (r(\tau; \bar{\tau}))^3 \lambda_r(\tau, r(\tau; \bar{\tau}))}{(\Xi(\tau; \bar{\tau}))^3} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \end{aligned}$$

where:

$$(\partial_r w)(\tau, r(\tau; \bar{\tau})) = \left(\frac{\partial w}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right) / \left(\frac{\partial r}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right)$$

Combining the terms on the right-hand side of (5.77) we obtain $\frac{\partial}{\partial \tau} \left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right] = 0$, whence (5.76) follows.

In order to prove the result if the derivative $\frac{\partial}{\partial \tau} \left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right]$ does not exist, we use the following argument. The weak formulation for the equations satisfied by $\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}}$ and $D(\tau; \bar{\tau})$ have the form:

$$\int \int \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} \frac{\partial \varphi}{\partial \tau}(\tau; \bar{\tau}) d\tau d\bar{\tau} = \int \int A_1(\tau; \bar{\tau}) \varphi(\tau; \bar{\tau}) d\tau d\bar{\tau} \quad (5.78)$$

$$\int \int D(\tau; \bar{\tau}) \frac{\partial \varphi}{\partial \tau}(\tau; \bar{\tau}) d\tau d\bar{\tau} = \int \int A_2(\tau; \bar{\tau}) \varphi(\tau; \bar{\tau}) d\tau d\bar{\tau} \quad (5.79)$$

for any compactly supported test function φ . Moreover, the following identity holds:

$$A_1(\tau; \bar{\tau}) D(\tau; \bar{\tau}) + A_2(\tau; \bar{\tau}) \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} = 0 \quad (5.80)$$

Suppose that $\zeta_\varepsilon \in C^\infty(\mathbb{R})$ is a mollifier, compactly supported, nonnegative, which satisfies $\zeta_\varepsilon(s) = \zeta_\varepsilon(-s)$ and converges to a Dirac mass in the sense of measures. We will assume also that $s\zeta_\varepsilon(s) \leq 0$. We take the test function $\varphi = \zeta_\varepsilon * [D \cdot \psi]$ in (5.78) and $\varphi = \zeta_\varepsilon * \left[\frac{\partial r}{\partial \bar{\tau}} \cdot \psi \right]$, where $*$ denotes the convolution in the variable τ and $\psi \in C^\infty$ is a compactly supported test function. We then obtain, exchanging the roles of the variables τ and ξ and using the symmetry properties of ζ_ε that:

$$\begin{aligned} & \int \int \int \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\xi; \bar{\tau}) \zeta'_\varepsilon(\tau - \xi) [\psi(\xi; \bar{\tau}) - \psi(\tau; \bar{\tau})] d\tau d\bar{\tau} d\xi \\ &= \int \int \int \zeta'_\varepsilon(\tau - \xi) \psi(\xi; \bar{\tau}) \left[A_1(\tau; \bar{\tau}) D(\xi; \bar{\tau}) + A_2(\tau; \bar{\tau}) \frac{\partial r(\xi; \bar{\tau})}{\partial \bar{\tau}} \right] d\tau d\bar{\tau} d\xi \end{aligned} \quad (5.81)$$

Taking the limit $\varepsilon \rightarrow 0$, using Lebesgue's dominated convergence Theorem and using (5.80) we obtain that the right-hand side of (5.81) converges to zero. On the other hand $\zeta'_\varepsilon(\tau - \cdot) [\psi(\cdot; \bar{\tau}) - \psi(\tau; \bar{\tau})]$ converges to a Dirac mass multiplied by $\frac{\partial \varphi}{\partial \tau}(\tau; \bar{\tau})$ at the point $\xi = \tau$. Using again Lebesgue's Theorem we obtain:

$$\int \int \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \frac{\partial \varphi}{\partial \tau}(\tau; \bar{\tau}) d\tau d\bar{\tau} = 0$$

This is the weak formulation of the identity (5.76) whence the result follows. \blacksquare

5.8 Relation between the solutions of (5.21)-(5.24) in the sense of characteristics and the weak solutions of the problem.

We now prove that it is possible to obtain a solution of (2.5)-(2.8), (2.16), (2.17) in the sense of Definition 13 with the form (3.2) by glueing a self-similar

solution with the form given in Theorem 22 with another solution with the form (5.10) for $r > R_+(t)$ and v_1, B_1 as in (5.20) with w, D satisfying (5.21), (5.22), (5.24) in the sense of characteristics. We will denote as Ω_{t_0} the domain $\{(t, r) : t > t_0, r > R_+(t)\}$.

Proposition 32 *Suppose that the functions $w \in W_{\text{loc}}^{1,\infty}(\Omega_{t_0})$, $D \in W_{\text{loc}}^{1,\infty}(\Omega_{t_0})$ solve the equations (5.21), (5.22) for a.e. $(t, r) \in \Omega_{t_0}$, with boundary conditions (5.24), where $R_+(t)$ is as in Proposition 23. Let us define v_1, B_1 for $r > R_+(t)$, $t \geq t_0$ by means of (5.20). Suppose that $\lambda, \mu \in W^{1,\infty}(\{t > t_0\})$ are defined as in Lemma 2. Let us assume that we extend v_1, B_1, λ, μ to $r \leq r_+(\tau)$ as in Theorem 22. We define also v_2 as in Theorem 22 for all $r > 0$. We define B_2 as in Theorem 22 for $r \leq R_+(t)$ and we extend B_2 as zero for $r > R_+(t)$. Then, the measure ζ defined as in (3.2), as well as the fields λ, μ in (3.8), (3.9) solve (2.5)-(2.8), (2.16), (2.17) in the sense of Definition 13 for $t > t_0$.*

The proof of Proposition 32 will be the content of the rest of this subsection. We first need to rewrite some of the terms appearing in (3.19), (3.20) and Definition 13.

Lemma 33 *Suppose that Δ is as in (3.12) with λ, μ satisfying (5.14) and $r(\tau; \bar{\tau}), w(\tau; \bar{\tau}) \in W_{\text{loc}}^{1,\infty}(\mathcal{U}(T))$ solve (5.30), (5.31). Then, the following identity holds:*

$$\Delta(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) e^{-\tau} = \frac{d}{d\tau} (\varphi(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau}))) \quad \text{a.e. } (\tau; \bar{\tau}) \in \mathcal{U}(T) \quad (5.82)$$

where τ is as in (5.1).

Proof. We first compute the right-hand side of (5.82):

$$\begin{aligned} & \frac{d}{d\tau} (\varphi(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau}))) \\ &= \varphi_t(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) e^{-\tau} + \varphi_r(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) \frac{\partial r(\tau; \bar{\tau})}{\partial \tau} \\ &+ \varphi_w(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) \frac{\partial w(\tau; \bar{\tau})}{\partial \tau} \end{aligned}$$

Using (5.30), (5.31) we obtain:

$$\begin{aligned} & \frac{d}{d\tau} (\varphi(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau}))) \quad (5.83) \\ &= \varphi_t(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) e^{-\tau} \\ &+ \varphi_r(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau})) - 2\lambda(\tau, r(\tau; \bar{\tau}))} w(\tau; \bar{\tau}) r(\tau; \bar{\tau})}{\Xi(\tau; \bar{\tau})} \\ &+ \varphi_w(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) \left[-\frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau}))} (e^{2\lambda(\tau, r(\tau; \bar{\tau}))} - 1)}{2\Xi(\tau; \bar{\tau})} \left[\frac{1}{(r(\tau; \bar{\tau}))^2} + 2e^{-2\lambda(\tau, r(\tau; \bar{\tau}))} (w(\tau; \bar{\tau}))^2 \right] \right. \\ &\left. + \frac{e^{\bar{\mu}(\tau, r(\tau; \bar{\tau}))}}{(r(\tau; \bar{\tau}))^2 \Xi(\tau; \bar{\tau})} \right] \end{aligned}$$

On the other hand expanding the derivatives ∂_r and $\partial_{\bar{v}}$ in (3.12) we obtain:

$$\Delta(t, r, \bar{v}) = \partial_t \varphi(t, r, \bar{v}) + e^{\mu-2\lambda} \frac{\bar{v}}{\tilde{E}} \varphi_r - \left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\tilde{E}} + e^\mu \mu_r \tilde{E} - e^\mu \frac{1}{r^3 \tilde{E}} \right) \varphi_{\bar{v}} \quad (5.84)$$

where we have used that

$$\partial_r \left(e^{\mu-2\lambda} \frac{\bar{v}}{\tilde{E}} \right) - \partial_{\bar{v}} \left(\left(-\frac{\lambda_r e^{\mu-2\lambda} \bar{v}^2}{\tilde{E}} + e^\mu \mu_r \tilde{E} - e^\mu \frac{1}{r^3 \tilde{E}} \right) \right) = 0$$

something that follows from an explicit computation. Using (5.14) we can eliminate the derivatives λ_r , μ_r from (5.84). Moreover, using also (5.2) we arrive, after some computations, at:

$$\begin{aligned} e^{-\tau} \Delta(t, r, \bar{v}) & \quad (5.85) \\ = e^{-\tau} \partial_t \varphi(t, r, \bar{v}) + e^{\bar{\mu}-2\lambda} \frac{\bar{v}}{\tilde{E}} \varphi_r - \left(-\frac{(1-e^{2\lambda}) e^{\bar{\mu}-2\lambda} \bar{v}^2}{2r \tilde{E}} + e^{\bar{\mu}} (e^{2\lambda} - 1) \frac{\tilde{E}}{2r} - e^{\bar{\mu}} \frac{1}{r^3 \tilde{E}} \right) \varphi_{\bar{v}} \end{aligned}$$

The identity (5.82) then follows combining (5.83) and (5.85). ■

We now prove the following:

Lemma 34 *Suppose that Δ , λ , μ are as in Lemma 33. Suppose also that $(r, w, D) \in \mathcal{X}_{L,T}$ and let us assume that D , w and \bar{D} , \bar{w} are related as in (5.62), (5.63). Let R_+ , r_+ as in Proposition 23. Then:*

$$\begin{aligned} & \int_{t_0}^0 \int_{R_+(t)}^\infty D(t, r) \Delta(t, r, \bar{w}(t, r)) dr dt \\ &= \int_T^\infty \frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} D(\bar{\tau}; \bar{\tau}) \varphi(\bar{t}, r(\bar{\tau}; \bar{\tau}), w(\bar{\tau}; \bar{\tau})) d\bar{\tau} \\ & - \int_{R_+(T)}^\infty \bar{D}(T, r) \varphi(t, r, \bar{w}(T, r)) dr \end{aligned} \quad (5.86)$$

Proof. Our goal is to compute the left-hand side of (5.86). To this end, we replace the variable of integration t by τ using (5.1). Due to Lemma 27 we can also replace the variable r by $\bar{\tau}$, using the change of variable $r = r(\tau; \bar{\tau})$. Then, using that $\bar{w}(t, r(\tau; \bar{\tau}(\tau, r))) = w(\tau; \bar{\tau})$ with $\bar{\tau}(\tau, r)$ as in Lemma 27 we obtain:

$$\begin{aligned} K & \equiv \int_{t_0}^0 \int_{R_+(t)}^\infty D(t, r) \Delta(t, r, w(t, r)) dr dt \\ &= \int_{\log(\frac{1}{(-t_0)})}^\infty \int_\tau^\infty D(\tau; \bar{\tau}) \Delta(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) e^{-\tau} \frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} d\bar{\tau} d\tau \end{aligned}$$

Using Lemma 33 we then obtain:

$$K = \int_{\log(\frac{1}{(-t_0)})}^\infty \int_\tau^\infty \left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right] \frac{d}{d\tau} (\varphi(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau}))) d\bar{\tau} d\tau$$

Using then Lemma 31 which implies that $\left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right]$ is constant almost everywhere we obtain:

$$K = \int_{\log\left(\frac{1}{(-t_0)}\right)}^{\infty} d\bar{\tau} \frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} D(\bar{\tau}; \bar{\tau}) \varphi(\bar{t}, r(\bar{\tau}; \bar{\tau}), w(\bar{\tau}; \bar{\tau})) \quad (5.87)$$

$$- \int_{\log\left(\frac{1}{(-t_0)}\right)}^{\infty} d\bar{\tau} \left[\frac{\partial r(\tau; \bar{\tau})}{\partial \bar{\tau}} D(\tau; \bar{\tau}) \right] \varphi(t, r(\tau; \bar{\tau}), w(\tau; \bar{\tau})) \Big|_{\tau=\log\left(\frac{1}{(-t_0)}\right)}$$

The last term in (5.87) can be transformed into an integral on $r > r_+(T)$ using again the change of variables $r = r(\tau; \bar{\tau})$. We then obtain (5.86). ■

We now need the following result, which is basically an auxiliary computation.

Lemma 35 *Suppose that $\bar{w}_k, \bar{D}_k \in C^1(\{t_0 < t < 0, y_0\sqrt{-t} < r < R_+(t)\})$, $k = 1, 2$ satisfy:*

$$\partial_t \bar{w}_k(t, r) + a_k(t, r, \bar{w}_k(t, r)) \partial_r \bar{w}_k(t, r) = Q_1(t, r, \bar{w}_k(t, r)) \quad (5.88)$$

$$\partial_t \bar{D}_k(t, r) + a_k(t, r, \bar{w}_k(t, r)) \partial_r \bar{D}_k(t, r) = Q_2(t, r, \bar{w}_k(t, r)) \bar{D}_k(t, r) \quad (5.89)$$

where

$$Q_1(t, r, \bar{w}_k) = \frac{e^\mu}{r \tilde{E}_k} \left(\frac{1}{r^2} - \frac{(e^{2\lambda} - 1)}{2} \left(\frac{1}{r^2} + 2e^{-2\lambda} \bar{w}_k^2 \right) \right) \quad (5.90)$$

$$Q_2(t, r, \bar{w}_k) = -\frac{e^{\mu-2\lambda}}{\tilde{E}} \left(\frac{\partial_r \bar{w}_k}{\tilde{E}_k^2 r^2} - \frac{\lambda_r \bar{w}_k}{\tilde{E}_k^2 r^2} + (\mu_r - \lambda_r) \bar{w}_k + \frac{\bar{w}_k}{r^3 \tilde{E}_k^2} \right) \quad (5.91)$$

$$Q_3(t, r, \bar{w}_k) = \left(\frac{\lambda_r e^{\mu-2\lambda} \bar{w}_k^2}{\tilde{E}} - e^\mu \mu_r \tilde{E} + e^\mu \frac{1}{r^3 \tilde{E}} \right) \quad (5.92)$$

$$a_k(t, r, \bar{w}_k) = e^{\mu-2\lambda} \frac{\bar{w}_k}{\tilde{E}_k} \quad (5.93)$$

where $\lambda, \mu \in C^1(\{t_0 < t < 0, y_0\sqrt{-t} < r < R_+(t)\})$. Then:

$$\frac{d}{dt} (\varphi(t, r, \bar{w}_k(t, r))) + \frac{d}{dr} (a(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r))) = J_k(t, r) + \Delta_k(t, r) \quad (5.94)$$

with:

$$J_k(t, r) = [Q_1(t, r, \bar{w}_k(t, r)) - Q_3(t, r, \bar{w}_k(t, r))] (\partial_w \varphi)(t, r, \bar{w}_k(t, r)) \quad (5.95)$$

$$+ [a_w(t, r, \bar{w}_k(t, r)) (\partial_r \bar{w}_k)(t, r) - (\partial_w Q_3)(t, r, \bar{w}_k(t, r))] \varphi(t, r, \bar{w}_k(t, r))$$

and Δ_k is defined for each $k = 1, 2$ as $\Delta_k(t, r) = \Delta(t, r, \bar{w}_k(t, r))$ where Δ is as in (3.20).

Remark 36 *Notice that we do not require λ, μ to solve the equations for the fields. They can be general arbitrary fields.*

Proof. Using (5.88) we obtain:

$$\begin{aligned}
& \frac{d}{dt} (\varphi(t, r, \bar{w}_k(t, r))) + \frac{d}{dr} (a_k(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r))) \\
&= (\partial_t \varphi)(t, r, \bar{w}_k(t, r)) + (\partial_w \varphi)(t, r, \bar{w}_k(t, r)) Q_1(t, r, \bar{w}_k(t, r)) \\
&+ a_{k,r}(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r)) + a_{k,w}(t, r, \bar{w}_k(t, r)) (\partial_r \bar{w}_k)(t, r) \varphi(t, r, \bar{w}_k(t, r)) \\
&+ a_k(t, r, \bar{w}_k(t, r)) (\partial_r \varphi)(t, r, \bar{w}_k(t, r)) \tag{5.96}
\end{aligned}$$

We remark now that the definition of Δ in (3.20) yields:

$$\begin{aligned}
\Delta_k(t, r) &= \partial_t \varphi(t, r, \bar{w}_k) + \partial_r (a_k(t, r, \bar{w}_k) \varphi) + \partial_w (Q_3(t, r, \bar{w}_k) \varphi) \\
&= \partial_t \varphi(t, r, \bar{w}_k) + (\partial_r a_k)(t, r, \bar{w}_k) \varphi(t, r, \bar{w}_k) + a(t, r, \bar{w}_k) (\partial_r \varphi)(t, r, \bar{w}_k) + \partial_w (Q_3(t, r, \bar{w}_k) \varphi)
\end{aligned}$$

Combining this identity with (5.96) we obtain:

$$\begin{aligned}
& \frac{d}{dt} (\varphi(t, r, \bar{w}_k(t, r))) + \frac{d}{dr} (a_k(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r))) \\
&= (\partial_w \varphi)(t, r, \bar{w}_k(t, r)) Q_1(t, r, \bar{w}_k(t, r)) \\
&+ a_{k,r}(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r)) + a_{k,w}(t, r, \bar{w}_k(t, r)) (\partial_r \bar{w}_k)(t, r) \varphi(t, r, \bar{w}_k(t, r)) \\
&+ \Delta_k(t, r) - (\partial_w (Q_3 \varphi))(t, r, \bar{w}_k(t, r)) - (\partial_r a_k)(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r))
\end{aligned}$$

and, after some computations we arrive at:

$$\begin{aligned}
& \frac{d}{dt} (\varphi(t, r, \bar{w}_k(t, r))) + \frac{d}{dr} (a_k(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r))) \\
&= [Q_1(t, r, \bar{w}_k(t, r)) - Q_3(t, r, \bar{w}_k(t, r))] (\partial_w \varphi)(t, r, \bar{w}_k(t, r)) \\
&+ [a_{k,w}(t, r, \bar{w}_k(t, r)) (\partial_r \bar{w}_k)(t, r) - (\partial_w Q_3)(t, r, \bar{w}_k(t, r))] \varphi(t, r, \bar{w}_k(t, r)) \\
&+ \Delta_k(t, r)
\end{aligned}$$

Using then (5.95) we obtain (5.94). ■

We can now compute a simpler form for $J_k(t, r)$.

Lemma 37 *Suppose that the conditions of Lemma 35 are satisfied. Let us assume also that the field equations (5.28) hold, with v_k , B_k and \bar{w}_k , \bar{D}_k related by means of (5.20). Then, the following identity holds:*

$$J_k(t, r) + Q_2(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r)) = 0 \tag{5.97}$$

where J_k is as (5.95) and Q_2 is as in (5.91).

Proof. Using (5.90), (5.92) we obtain:

$$\begin{aligned}
& Q_1(t, r, \bar{w}_k) - Q_3(t, r, \bar{w}_k) \\
&= \frac{e^\mu}{r \tilde{E}} \left(-\frac{(e^{2\lambda} - 1)}{2} \left(\frac{1}{r^2} + 2e^{-2\lambda} \bar{w}_k^2 \right) - r \lambda_r e^{-2\lambda} \bar{w}_k^2 + r \mu_r \tilde{E}^2 \right)
\end{aligned}$$

Using now (5.14) and (5.20) to eliminate the derivatives λ_r , μ_r we obtain, after some computations:

$$\begin{aligned} & Q_1(t, r, \bar{w}_k) - Q_3(t, r, \bar{w}_k) \\ &= \frac{e^\mu}{\bar{E}} \left[- \left(\frac{(e^{2\lambda} - 1)}{2r} \left(\frac{1}{r^2} + 2e^{-2\lambda} \bar{w}_k^2 \right) \right) + \left(\frac{e^{2\lambda} - 1}{2r} \right) \left(2\bar{w}_k^2 e^{-2\lambda} + \frac{1}{r^2} \right) \right] \\ &= 0 \end{aligned} \quad (5.98)$$

We now compute $[a_{k,w}(t, r, \bar{w}(t, r))(\partial_r \bar{w}_k)(t, r) - (\partial_w Q_3)(t, r, \bar{w}_k(t, r)) + Q_2(t, r, \bar{w}_k(t, r))]$. Using (5.91)-(5.93) we arrive at:

$$\begin{aligned} & [a_{k,w}(t, r, \bar{w}_k(t, r))(\partial_r \bar{w}_k)(t, r) - (\partial_w Q_3)(t, r, \bar{w}_k(t, r)) + Q_2(t, r, \bar{w}_k(t, r))] \\ &= \frac{\lambda_r e^{\mu-4\lambda} \bar{w}_k^3}{\sqrt{(\bar{w}_k^2 e^{-2\lambda} + \frac{1}{r^2})^3}} - \frac{e^{\mu-4\lambda} \bar{w}_k^3 \lambda_r}{\bar{E}_k^3} = 0 \end{aligned} \quad (5.99)$$

Combining (5.95), (5.98), (5.99) we obtain (5.97). ■

We now combine Lemmas 35, 37 in order to rewrite the integrals $\int_{t_0}^0 \int_0^{R_+(t)} \bar{D}_k(t, r) \Delta_k(t, r, \bar{w}_k(t, r)) dr dt$ in terms of initial and boundary values.

Lemma 38 *Suppose that $\bar{w}_k, \bar{D}_k, \Delta_k$ are as in Lemma 35 and that the conditions of Lemmas 35, 37 hold. Then, the following identity holds:*

$$\begin{aligned} & \int_{t_0}^0 \int_0^{R_+(t)} \bar{D}_k(t, r) \Delta_k(t, r) dr dt \\ &= - \int_0^{R_+(t_0)} \bar{D}_k(t_0, r) \varphi(t_0, r, \bar{w}_k(t_0, r)) dr \\ & - \int_{t_0}^0 D_k(t, R_{k,+}(t)) \varphi(t, R_+(t), w_k(t, R_+(t))) \frac{dR_{k,+}(t)}{dt} dt \\ & + \int_{t_0}^0 D_k(t, R_{k,+}(t)) a_k(t, R_{k,+}(t), w_k(t, R_{k,+}(t))) \varphi(t, R_{k,+}(t), w_k(t, R_{k,+}(t))) dt \end{aligned} \quad (5.100)$$

for $k = 1, 2$, where $R_{k,+}(t) = R_{\max}$ if $k = 1$ and $R_{k,+}(t) = R_+(t)$ if $k = 2$.

Proof. Using (5.94) we can rewrite the left-hand side of (5.100) as:

$$\begin{aligned}
& \int_{t_0}^0 \int_0^{R_+(t)} \bar{D}_k(t, r) \Delta_k(t, r) dr dt \\
&= \int_{t_0}^0 \frac{d}{dt} \left(\int_0^{R_{k,+}(t)} \bar{D}_k(t, r) \varphi(t, r, \bar{w}_k(t, r)) dr \right) dt \\
&\quad - \int_{t_0}^0 \bar{D}_k(t, R_+(t)) \varphi(t, R_{k,+}(t), \bar{w}_k(t, R_{k,+}(t))) \frac{dR_{k,+}(t)}{dt} dt \\
&\quad - \int_{t_0}^0 \int_0^{R_{k,+}(t)} (\partial_t \bar{D}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r)) dr dt \\
&\quad + \int_{t_0}^0 \int_0^{R_{k,+}(t)} \bar{D}_k(t, r) \frac{d}{dr} (a_k(t, r, \bar{w}_k(t, r)) \varphi(t, r, \bar{w}_k(t, r))) dr dt \\
&\quad - \int_{t_0}^0 \int_0^{R_{k,+}(t)} \bar{D}_k(t, r) J_k(t, r) dr dt
\end{aligned}$$

Integrating by parts and using (5.89) we obtain:

$$\begin{aligned}
& \int_{t_0}^0 \int_{R_+(t)}^\infty \bar{D}_k(t, r) \Delta_k(t, r) dr dt \\
&= - \int_0^{R_+(t_0)} \bar{D}_k(t_0, r) \varphi(t_0, r, \bar{w}_k(t_0, r)) dr \\
&\quad - \int_{t_0}^0 \bar{D}_k(t, R_{k,+}(t)) \varphi(t, R_{k,+}(t), \bar{w}_k(t, R_{k,+}(t))) \frac{dR_{k,+}(t)}{dt} dt \\
&\quad + \int_{t_0}^0 \bar{D}_k(t, R_{k,+}(t)) a(t, R_{k,+}(t), \bar{w}_k(t, R_{k,+}(t))) \varphi(t, R_{k,+}(t), \bar{w}_k(t, R_{k,+}(t))) dt \\
&\quad - \int_{t_0}^0 \int_0^{R_+(t)} Q_2(t, r, \bar{w}_k(t, r)) \bar{D}_k(t, r) \varphi(t, r, \bar{w}_k(t, r)) dr dt \\
&\quad - \int_{t_0}^0 \int_0^{R_{k,+}(t)} \bar{D}_k(t, r) J_k(t, r) dr dt
\end{aligned}$$

Using then (5.97) we obtain (5.100) and the result follows. ■

We can finally conclude the proof of Proposition 32.

End of the proof of Proposition 32. Given a test function φ we can split it into the sum of two pieces φ_1, φ_2 where the support of φ_1 is in the region $r \leq \frac{2R_+(t)}{3}$ and the support of φ_2 is in the set $r \geq \frac{R_+(t)}{2}$. Using Theorem 22 we obtain that the corresponding formula (3.19) holds for φ_1 . We then need to check (3.19) for φ_2 .

The continuity of the function $\Psi(t, r, \bar{v})$ in $S \cap \{r \geq R_+(t)\}$ follows from the properties of the functions v_1, B_1, λ, μ . On the other hand, using Lemma 34

we obtain:

$$\begin{aligned}
& \int \int_{S \cap \{r > R_+(t)\}} \tilde{\zeta}(t, r, \bar{v}) \Delta(t, r, \bar{v}) dr d\bar{v} dt \\
&= \int_T^\infty \frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} D(\bar{\tau}; \bar{\tau}) \varphi(\bar{t}, r(\bar{\tau}; \bar{\tau}), \bar{w}(\bar{\tau}; \bar{\tau})) d\bar{\tau} \\
&\quad - \int_{r_+(T)}^\infty dr \bar{D}(T, r) \varphi(t, r, \bar{w}(r, T))
\end{aligned} \tag{5.101}$$

On the other hand, using the fact that $\Delta(t, r, \bar{v})$ in $S \cap \{r < R_+(t)\}$ can be written as the sum $\sum_{k=1}^2 \Delta_k(t, r)$, as well as Lemma 38 we arrive at:

$$\begin{aligned}
& \int \int_{S \cap \{r < R_+(t)\}} \tilde{\zeta}(t, r, \bar{v}) \Delta(t, r, \bar{v}) dr d\bar{v} dt \\
&= - \sum_{k=1}^2 \int_0^{R_+(t_0)} \bar{D}_k(t_0, r) \varphi(t_0, r, \bar{w}_k(t_0, r)) dr \\
&\quad - \sum_{k=1}^2 \int_{t_0}^0 \bar{D}_k(t, R_{k,+}(t)) \varphi(t, R_+(t), \bar{w}_k(t, R_+(t))) \frac{dR_{k,+}(t)}{dt} dt \\
&\quad + \sum_{k=1}^2 \int_{t_0}^0 \bar{D}_k(t, R_{k,+}(t)) a_k(t, R_{k,+}(t), \bar{w}_k(t, R_{k,+}(t))) \varphi(t, R_{k,+}(t), \bar{w}_k(t, R_{k,+}(t))) dt
\end{aligned} \tag{5.102}$$

Adding (5.101), (5.102) we obtain the cancellation of several terms. Indeed, the terms with $k = 2$ in (5.102) cancel out due to the definition of the function $R_+(t) = R_{2,+}(t)$. On the other hand, the terms with $k = 1$ along the boundary $r = R_+(t)$ can be computed as follows. We have:

$$r(\bar{\tau}; \bar{\tau}) = r_+(\bar{\tau})$$

Differentiating this formula we obtain:

$$\frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} + \frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} = \frac{dr_+(\bar{\tau})}{d\bar{\tau}}$$

Using that $\frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} = a_1$ we obtain:

$$a_1(R_+(\bar{t}), w_k(R_+(\bar{t}), \bar{t}), \bar{t}) + \frac{\partial r(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} = \frac{dr_+(\bar{\tau})}{d\bar{\tau}}$$

It then follows that the sum of the terms in (5.101), (5.102) cancel out. Therefore:

$$\begin{aligned}
& \int \int_S \tilde{\zeta}(t, r, \bar{v}) \Delta(t, r, \bar{v}) dr d\bar{v} dt \\
&= - \int_{R_+(t_0)}^\infty D(t_0, r) \varphi(t_0, r, w(t_0, r)) dr - \sum_{k=1}^2 \int_0^{R_+(t_0)} D_k(t_0, r) \varphi(t_0, r, w_k(t_0, r)) dr
\end{aligned}$$

and Proposition 32 follows. ■

Our next goal is to prove the following result.

Proposition 39 *There exists $t_0 < 0$ and functions $(r, w, D) \in \mathcal{X}_{L,T}$ with $L > L_0$ and $T > T_0$, where L_0, T_0 are sufficiently large, as well as functions λ, μ defined as in Lemma 2 with ρ and p defined as in (2.9), (2.10) where the functions (r, w, D) are a solution of (5.21)-(5.24) in the sense of characteristics, as in Definition 30.*

The proof of Proposition 39 will be carried out by means of a fixed point argument. In order to formulate this argument in a precise manner, it is convenient to reformulate the equations (5.11), (5.12) in a way, that will allow to derive suitable regularity estimates for some of the functions involved.

5.9 Introducing a topology in $\mathcal{Y}_{L,T,a}$.

The next goal is to define a suitable weak topology in the set of functions $\mathcal{Y}_{L,T,a}$. Suppose that $\tau^* \geq T$. For any $\varepsilon > 0$ and $\varphi \in C_0^\infty[r_+(\tau^*), \infty)$, we define a neighbourhood of a function $\bar{D}^{(\infty)}$ in $\mathcal{A}_{\tau^*} = L^\infty[r_+(\tau^*), \infty)$ as the set of functions \bar{D} such that \bar{D} satisfies (5.55) and

$$\left| \int_{[r_+(\tau^*), \infty)} \varphi(r) [\bar{D}(\tau^*, r) - \bar{D}^{(\infty)}(\tau^*, r)] dr \right| < \varepsilon \quad (5.103)$$

Classical results show that the topology defined by means of these functionals is metrizable (cf. [3]). In particular convergence can be characterized by sequential limits. Notice that the spaces (and the corresponding metric) change with τ^* because the domain of the functions depends on τ^* . In any case, we can define a topology for the functions \bar{D} . We will denote as d_{τ^*} the metric associated to the weak topology defined in \mathcal{A}_{τ^*} . We will assume in all the following that the space \mathcal{A}_{τ^*} is endowed with the topology generated by the functionals (5.103). We then introduce a topology for the functions \bar{D} defined in a suitable space termed as:

$$C([T, \infty) : \mathcal{A}_{\tau^*}) \quad (5.104)$$

where by this space we mean a set of functions $\bar{D}(\tau^*, \cdot) \in \mathcal{A}_{\tau^*}$. Notice that these functions can be also thought of as functions defined in the domain $\{T \leq \tau^* < \infty, r_+(\tau^*) < r < \infty\}$. The metric of this space is given by means of:

$$\text{dist}(\bar{D}_1^{(1)}, \bar{D}_1^{(2)}) = \sup_{\tau^* \in [T, \infty)} d_{\tau^*}(\bar{D}_1^{(1)}(\tau^*, \cdot), \bar{D}_1^{(2)}(\tau^*, \cdot)) \quad (5.105)$$

Notice that there is not any problem defining the integrals in (5.103) because the test functions are compactly supported. The topology generated by these functionals is metrizable due to (5.55).

We need to obtain a criteria of compactness in $\mathcal{Y}_{L,T,a}$. To this end, we introduce the following definition.

Definition 40 *We will say that the set of functions $\mathcal{F} \subset \mathcal{Y}_{L,T,a}$ is equicontinuous if for any $\varepsilon > 0$ and any test function $\varphi \in C_0^\infty([0, \infty))$ there exists*

$\delta = \delta(\varepsilon, \varphi) > 0$ such that, for any $\tau_1, \tau_2 \in [T, \infty)$ satisfying $|\tau_1 - \tau_2| \leq \delta$ and any $(r, w, D) \in \mathcal{F}$ we have:

$$\left| \int_{[r_+(\tau_1), \infty)} \bar{D}(\tau_1, r) \varphi(r - r_+(\tau_1)) dr - \int_{[r_+(\tau_2), \infty)} \bar{D}(\tau_2, r) \varphi(r - r_+(\tau_2)) dr \right| \leq \varepsilon$$

where \bar{D} is the function defined in Lemma 27.

Remark 41 The functions $\bar{D}(\cdot, \tau)$ are defined in the domains $(r_+(\tau), \infty)$. A translation has been made in order to bring the spaces to a fixed domain, and use test functions φ defined in a fixed domain to measure their distance.

We have the following result:

Proposition 42 Suppose that $\mathcal{F} \subset \mathcal{Y}_{L,T,a}$ is equicontinuous in the sense of Definition 40. Then, there exists a subsequence $\{D_n\}$ such that the corresponding functions \bar{D}_n defined by means of Lemma 27 converge in the space $C([T, \infty) : \mathcal{A}_{\tau^*})$ to some function $\bar{D} \in L^\infty((\tau, r) : r \geq r_+(\tau), \tau \geq T)$.

Proof. The result is just a consequence of the classical Arzela-Ascoli Theorem in arbitrary metric spaces (cf. ([9]), ([11])). ■

We now prove the following:

Lemma 43 Suppose that $\mathcal{F} \subset \mathcal{Y}_{L,T,a}$ is equicontinuous in the sense of Definition 40. Then \mathcal{F} is compact in the topological space $\mathcal{X}_{L,T}$ endowed with the topology generated by (5.103), (5.105).

Proof. We first prove that the space $\mathcal{Y}_{L,T,a}$ is closed with this topology. To this end, we just notice that (5.70) is preserved under limits in this topology. The topology defined in $\mathcal{Y}_{L,T,a}$ is metrizable. Therefore, we can restrict our analysis to the convergence of sequences. The conservation of the inequality (5.55) follows from a general argument yielding preservation of inequalities for weak topologies. Indeed, given a test function $\varphi \in C_0^\infty(\{T \leq \tau \leq \bar{\tau}\})$ suppose that we have a sequence $\{f_n\} \subset L^\infty(\{r \geq r_+(\tau), \tau \geq T\})$ such that $\int f_n \varphi \rightarrow \int f_\infty \varphi$ as $n \rightarrow \infty$, with $f_\infty \in L^\infty(\{r \geq r_+(\tau), \tau \geq T\})$. Suppose that the functions f_n satisfy the inequalities $f_n \leq \psi$ for a.e. $(\tau, r) \in \{r \geq r_+(\tau), \tau \geq T\}$. Choosing any test function $\varphi \in C_0^\infty(\{r \geq r_+(\tau), \tau \geq T\})$ such that $\varphi \geq 0$, it then follows that:

$$\int f_n \varphi d\tau d\bar{\tau} \leq \int \psi \varphi d\tau d\bar{\tau}$$

and taking the limit in this inequality we obtain:

$$\int f_\infty \varphi d\tau d\bar{\tau} \leq \int \psi \varphi d\tau d\bar{\tau}$$

since φ is an arbitrary, compactly supported, nonnegative function we then obtain $f_\infty \leq \psi$, a.e. $(\tau, r) \in \{r \geq r_+(\tau), \tau \geq T\}$. ■

We will use the following auxiliary result.

Lemma 44 Given $(r, w, D) \in \mathcal{X}_{L,T}$ we define \bar{w}, \bar{D} as in Lemma 27. We then define the fields $\bar{\mu}, \lambda$ for $r \geq r_+(\tau), \tau > T$, by means of the following ODE problem:

$$e^{-2\lambda} (2r\lambda_r - 1) + 1 = 8\pi^2 e^{-\lambda} \tilde{E} \bar{D} \quad , \quad r \geq r_+(\tau) \quad (5.106)$$

$$e^{-2\lambda} (2r\bar{\mu}_r + 1) - 1 = \frac{8\pi^2 e^{-\lambda}}{\tilde{E}} \bar{w}^2 \bar{D} \quad , \quad r \geq r_+(\tau) \quad (5.107)$$

$$\tilde{E} = \sqrt{w^2 e^{-2\lambda} + \frac{1}{r^2}} \quad (5.108)$$

$$\lambda \left(\tau, r_+(\tau)^+ \right) = \lambda \left(\tau, r_+(\tau)^- \right) \quad , \quad \bar{\mu} \left(\tau, r_+(\tau)^+ \right) = \bar{\mu} \left(\tau, r_+(\tau)^- \right) \quad (5.109)$$

where $\lambda \left(\tau, r_+(\tau)^- \right), \bar{\mu} \left(\tau, r_+(\tau)^- \right)$ are computed using (3.3), (3.4), Theorem 22. Given L , there exists $T_0 = T_0(L)$ such that, if $T \geq T_0$, the functions $\bar{\mu}, \lambda$ defined by means of (5.106)-(5.109) are defined for $\tau \geq T, r \geq r_+(\tau)$. Moreover, if we endow the space of functions $C(\{\tau \geq T, r \geq r_+(\tau)\})$ with the topology of uniform convergence in compact sets, the functions $\bar{\mu}, \lambda$ define a mapping:

$$\bar{\mu}, \lambda : \mathcal{Y}_{L,T,a} \rightarrow C(\{\tau \geq T, r \geq r_+(\tau)\}) \quad (5.110)$$

which is continuous if $\mathcal{Y}_{L,T,a}$ is endowed with the topology generated by means of the functionals (5.103).

Remark 45 Notice that the topology of $\mathcal{X}_{L,T}$ only provides weak convergence along lines of constant τ for the functions \bar{D} .

Proof. Local existence of solutions follows from standard ODE Theory. In order to show that the solution is defined for arbitrarily large values of r , we use the fact that, arguing as in the proof of Lemma 2, we can obtain the following representation formula for the fields $\bar{\mu}, \lambda$ for $r > r_+(\tau)$

$$\lambda[(r, w, D)](\tau, r) = \lambda(\tau, r) = \frac{1}{2} \log \left(\frac{r}{r - R_0(\tau, r)} \right) = \frac{1}{2} \log(r) - \frac{1}{2} \log(r - R_0(\tau, r)) \quad (5.111)$$

$$\begin{aligned} \bar{\mu}[(r, w, D)](\tau, r) &= \bar{\mu}(\tau, r) = \bar{\mu}(\tau, r_+(\tau)) \\ &+ \int_{r_+(\tau)}^r \frac{4\pi^2 \exp(-\lambda(\tau, \xi)) \bar{w}^2(\tau, \xi) \bar{D}(\tau, \xi)}{\tilde{E}(\tau, \xi)} \frac{d\xi}{\xi - R_0} \\ &+ \frac{1}{2} \int_{r_+(\tau)}^r \frac{d\xi}{(\xi - R_0(\tau, \xi))} - \frac{1}{2} \log \left(\frac{r}{r_+(\tau)} \right) \end{aligned} \quad (5.112)$$

where:

$$\begin{aligned} R_0(\tau, r) &= r_+(\tau) (1 - \exp(-2\lambda(\tau, r_+(\tau)))) \\ &+ \int_{r_+(\tau)}^r 8\pi^2 \tilde{E}(\tau, \xi) \exp(-\lambda(\tau, \xi)) \bar{D}(\tau, \xi) d\xi \quad , \end{aligned} \quad (5.113)$$

for $r \geq r_+(\tau)$ and:

$$\tilde{E}(\tau, \xi) = \tilde{E} = \sqrt{(\bar{w}(\tau, \xi))^2 e^{-2\lambda(\tau, \xi)} + \frac{1}{r^2}} \quad (5.114)$$

Notice that (5.106) implies that $\lambda(\tau, r) > 0$ for $r \geq r_+(\tau)$. We can then estimate the exponential terms in (5.112)-(5.114) by 1. We now use the fact that $0 \leq \bar{D}(\tau, r) \leq Ce^{-ar}$ (cf. (5.64)) to obtain that, if T_0 is sufficiently large the functions λ and $\bar{\mu}$ are defined for arbitrary values $r \geq r_+(\tau)$, $\tau \geq T \geq T_0$.

In order to prove the continuity of the mappings (5.110) we consider the difference of the fields $\bar{\mu}$, λ associated to functions (r_1, w_1, D_1) and (r_2, w_2, D_2) . Notice that the ODEs (5.106), (5.107) imply estimates for the derivatives of $\bar{\mu}$, λ . We also have uniform estimates for the derivatives of the function \bar{w} . Suppose that we consider the difference of the functions λ_1 , λ_2 associated to (r_1, w_1, D_1) and (r_2, w_2, D_2) respectively. We then obtain, for each $\tau \geq T$ the following estimate (using (5.111), (5.113), (5.114)):

$$\begin{aligned} |\lambda_1(\tau, r) - \lambda_2(\tau, r)| &\leq C \int_{r_+(\tau)}^r |\lambda_1(\tau, \xi) - \lambda_2(\tau, \xi)| d\xi \\ &\quad + \left| \int_{r_+(\tau)}^r \Psi(\tau, \xi) [\bar{D}_1(\tau, \xi) - \bar{D}_2(\tau, \xi)] d\xi \right| + \bar{\varepsilon} \end{aligned}$$

where Ψ is a continuous function with $|\Psi_r|$ bounded. The error $\bar{\varepsilon}$ is due to the differences of terms \bar{w}_1, \bar{w}_2 . It can be made arbitrarily small if (r_1, w_1, D_1) and (r_2, w_2, D_2) are close in the topology of $\mathcal{X}_{L,T}$. Due to Arzela-Ascoli, for any $\varepsilon > 0$ and any $\tau \geq T$, there exists a finite set of functions $\psi_1, \psi_2, \dots, \psi_L$ such that for any Ψ as above with $|\Psi_r| \leq A$, we have $\min_k \sup_{r \in [0, R]} |\Psi(\tau, r) - \psi_k(\tau, r)| \leq \varepsilon$. Using also the boundedness of \bar{D}_1, \bar{D}_2 in order to show that approximating the integral $\int_{r_+(\tau)}^r \Psi(\tau, \xi) [\bar{D}_1(\tau, \xi) - \bar{D}_2(\tau, \xi)] d\xi$ for a finite set of values r_1, r_2, \dots, r_M we obtain an approximation in the whole interval $[0, R]$, as well as the fact that \bar{D}_1, \bar{D}_2 are close in the weak topology, we obtain that the difference $|\lambda_1(\tau, r) - \lambda_2(\tau, r)|$ can be made arbitrarily small for $r \in [0, R]$. The difference $|\bar{\mu}_1(\tau, r) - \bar{\mu}_2(\tau, r)|$ can be estimated in a similar form.

We notice also that we can prove that the functions $\lambda, \bar{\mu}$ are continuous in the variable τ using a similar argument. Indeed, taking the difference of λ at two different times $\tau_1, \tau_2 \geq T$ we would obtain:

$$\begin{aligned} |\lambda(\tau_1, r) - \lambda(\tau_2, r)| &\leq C \int_{r_+(\tau_1)}^r |\lambda(\tau_1, \xi) - \lambda(\tau_2, \xi)| d\xi \\ &\quad + \left| \int_{r_+(\tau_2)}^r \Psi(\tau, \xi) [\bar{D}(\tau_1, \xi) - \bar{D}(\tau_2, \xi)] d\xi \right| + \bar{\varepsilon} \end{aligned}$$

where the error term contains contributions due to the differences of \bar{w}_1, \bar{w}_2 as well as the modulus of continuity of $r_+(\tau)$. Using the fact that $\bar{D} \in C([T, \infty); \mathcal{A}_{\tau^*})$ we obtain that λ is continuous in the variable τ . The continuity of $\bar{\mu}$ can be proved in a similar manner. ■

6 FIXED POINT ARGUMENT.

6.1 Definition of the operator \mathcal{T} .

We now define an operator in the space $\mathcal{Y}_{L,T,a}$ as follows. Given $\bar{D} \in \mathcal{Y}_{L,T,a}$ we define fields $\lambda, \bar{\mu}$ as in Lemma 44. We then define an operator \mathcal{T} as:

$$\mathcal{T} : \bar{D} \rightarrow \bar{D}_n \quad (6.1)$$

where the functions \bar{D}_n are defined as follows (n means new). We first define the functions r_n, w_n as:

$$\frac{\partial r_n(\tau; \bar{\tau})}{\partial \tau} = \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - 2\lambda(\tau, r_n(\tau; \bar{\tau}))} w_n(\tau; \bar{\tau}) r_n(\tau; \bar{\tau})}{\Xi_n(\tau; \bar{\tau})}, \quad r_n(\bar{\tau}; \bar{\tau}) = r_+(\bar{\tau}) \quad (6.2)$$

$$\begin{aligned} \frac{\partial w_n(\tau; \bar{\tau})}{\partial \tau} = & -\frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau}))} (e^{2\lambda(\tau, r_n(\tau; \bar{\tau}))} - 1)}{2\Xi_n(\tau; \bar{\tau})} \left[\frac{1}{(r_n(\tau; \bar{\tau}))^2} + 2e^{-2\lambda(\tau, r_n(\tau; \bar{\tau}))} (w_n(\tau; \bar{\tau}))^2 \right] \\ & + \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau}))}}{(r_n(\tau; \bar{\tau}))^2 \Xi_n(\tau; \bar{\tau})} \end{aligned} \quad (6.3)$$

$$w_n(\bar{\tau}; \bar{\tau}) = \frac{e^{\lambda(\tau, r_n(\bar{\tau}; \bar{\tau}))}}{(-\bar{t})} V_1 \left(\frac{r_+(\bar{\tau})}{(-\bar{t})} \right) \quad (6.4)$$

where:

$$\Xi_n(\tau; \bar{\tau}) = \sqrt{1 + e^{-2\lambda(r_n(\tau; \bar{\tau}), \tau)} (w_n(\tau; \bar{\tau}))^2 (r_n(\tau; \bar{\tau}))^2} \quad (6.5)$$

We then define D_n by means of:

$$\begin{aligned} & \frac{\partial D_n(\tau; \bar{\tau})}{\partial \tau} \\ = & - \left(\frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - 2\lambda(\tau, r_n(\tau; \bar{\tau}))} \cdot \left[\frac{\left(\frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right)}{\left(\frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right)} - \lambda_r(r_n(\tau; \bar{\tau}), \tau) w_n(\tau; \bar{\tau}) \right] r_n(\tau; \bar{\tau})}{(\Xi_n(\tau; \bar{\tau}))^3} \right. \\ & + \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - 2\lambda(\tau, r_n(\tau; \bar{\tau}))} r_n(\tau; \bar{\tau}) w_n(\tau; \bar{\tau}) [\bar{\mu}_r(\tau, r_n(\tau; \bar{\tau})) - \lambda_r(\tau, r_n(\tau; \bar{\tau}))]}{\Xi_n(\tau; \bar{\tau})} \\ & \left. + \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - 2\lambda(\tau, r_n(\tau; \bar{\tau}))} w_n(\tau; \bar{\tau}) D_n(\tau; \bar{\tau})}{\Xi_n(\tau; \bar{\tau})} \right) \\ D_n(\bar{\tau}; \bar{\tau}) = & (-\bar{t}) b_1 \left(\frac{r_+(\bar{\tau})}{(-\bar{t})} \right) \exp(\lambda(\tau, r_n(\tau; \bar{\tau}))) \end{aligned} \quad (6.6)$$

Notice that in these equations we have replaced $(\partial_r \bar{w}_n)(r(\tau, \bar{\tau}), \tau)$ by $\left(\frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right) / \left(\frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right)$.

We then construct a function \bar{D}_n taking as starting point the functions (r_n, w_n, D_n) , which will be shown to be in the space $\mathcal{X}_{L,T}$, by means of Lemma 27.

The main result that we will prove in the following is that the operator \mathcal{T} is well defined and it maps $\mathcal{Y}_{L,T,a}$ into itself compactly if we assume that L and T are sufficiently large.

Proposition 46 *There exists L_0 sufficiently large such that, for $L > L_0$ there exists $T_0 = T_0(L)$ such that for $T > T_0$ the operator \mathcal{T} defined as in (6.1) is well defined for any $D \in \mathcal{Y}_{L,T,a}$ and it maps $\mathcal{Y}_{L,T,a}$ into itself. The operator \mathcal{T} is compact.*

We first remark that the solvability of (6.2)-(6.4), (6.6) can be obtained using standard ODE arguments.

Lemma 47 *For any $\bar{D} \in \mathcal{Y}_{L,T,a}$ there exist $\delta > 0$ and a unique solution of (6.2)-(6.4), (6.6) defined in $\{\bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}\}$ which satisfy these equations a.e. with $\bar{\mu}$, λ as in Lemma 44. The following estimates hold:*

$$0 < c_0 \leq \left| \frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right| \leq c_1 \quad , \quad \bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}$$

$$\left| \frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right| \leq c_1 \quad , \quad \bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}$$

We assume that $(\partial_r w_n)(r(\tau, \bar{\tau}), \tau) = \left(\frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right) / \left(\frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right)$. Moreover $r_n(\tau; \bar{\tau}) \geq r_+(\bar{\tau})$ if $\max\{T, \bar{\tau} - \delta\} \leq \tau \leq \bar{\tau}$. The solutions can be extended as long as w_n , r_n remain bounded and $r_n(\tau, \bar{\tau}) > 0$, $\left| \frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right| > 0$.

Proof. Due to (5.106), (5.107) as well as the boundedness of λ we have that λ_r , $\bar{\mu}_r$ are uniformly bounded in $\{\tau \geq T, r \geq r_+(\tau)\}$. Therefore, the right-hand side of (6.2), (6.3) is Lipschitz for bounded values of r_n, w_n . Then, the functions r_n, w_n can be defined by means of (6.2)-(6.4) due to local ODE theory. Notice that the global boundedness of $r_+(\bar{\tau})$, $\frac{e^{\lambda(r_n(\bar{\tau}; \bar{\tau}), \tau)}}{(-t)} V_1 \left(\frac{r_+(\bar{\tau})}{(-t)} \right)$ for $\bar{\tau} \geq T$ imply that the time of existence of solutions δ can be chosen uniformly in $\bar{\tau}$. We can now rewrite (6.2)-(6.4) in integral form and differentiate with respect to $\bar{\tau}$. We can then obtain uniform estimates for $\left| \frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right|$ and $\left| \frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right|$ using Gronwall's Lemma. Moreover, differentiating the identity $r_n(\bar{\tau}; \bar{\tau}) = r_+(\bar{\tau})$, and using the fact that $\frac{dr_+(\bar{\tau})}{d\bar{\tau}}$ is small for T sufficiently large, and $\left| \frac{\partial r_n(\bar{\tau}; \bar{\tau})}{\partial \bar{\tau}} \right| \geq c_2 > 0$ for $\bar{\tau} \geq T$ and T large (due to (6.2) and (6.4)) it then follows that $c_0 \leq \left| \frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right|$ if $\bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}$ and δ is small.

We can then define $(\partial_r w_n)(r(\tau, \bar{\tau}), \tau)$ as $\left(\frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right) / \left(\frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right)$ for $\bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}$. Then (6.6) has the form $\frac{\partial D_n(\tau; \bar{\tau})}{\partial \bar{\tau}} = F(\tau; \bar{\tau}) D_n(\tau; \bar{\tau})$ where $F(\tau; \bar{\tau})$ is bounded for $\bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}$. We can then obtain $D_n(\tau; \bar{\tau})$ as $D_n(\tau; \bar{\tau}) = D_n(\bar{\tau}; \bar{\tau}) \exp \left(- \int_{\tau}^{\bar{\tau}} F(s; \bar{\tau}) ds \right)$. Notice that the function $D_n(\tau; \bar{\tau})$ solves (6.6) a.e. We remark also that the right-hand side of (6.2) is negative if $r_n > 0$ and $w_n < 0$. Due to (6.4) we have $w_n < 0$ if $\bar{\tau} \geq \tau \geq \max\{T, \bar{\tau} - \delta\}$ and $\delta > 0$ is small enough. It then follows that $r_n(\tau; \bar{\tau}) \geq r_+(\bar{\tau})$ if $\max\{T, \bar{\tau} - \delta\} \leq \tau \leq \bar{\tau}$ and $\delta > 0$ is small. The fact that the solution can be extended as long as w_n, r_n are bounded and $r_n(\tau, \bar{\tau}) > 0$, $\left| \frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}) \right| > 0$ follows from the form of the right-hand side of (6.2), (6.3), (6.6). ■

6.2 The operator \mathcal{T} maps $\mathcal{Y}_{L,T,a}$ into itself if L and T are large.

In order to prove Proposition 46 we need to derive estimates for the functions $\rho_n^*(\tau; \bar{\tau})$, $z_n(\tau; \bar{\tau})$ which will be defined by means of:

$$\rho_n^*(\tau; \bar{\tau}) = (r_n(\tau; \bar{\tau}) - \mathcal{R}(\tau - \bar{\tau})) \quad , \quad z_n(\tau; \bar{\tau}) = (w_n(\tau; \bar{\tau}) - \mathcal{W}(\tau - \bar{\tau})) \quad , \quad \bar{\tau} \geq \tau \geq T \quad (6.7)$$

(cf. (5.57)). As a first step we obtain estimates for the difference between the functions $\lambda(r_n(\tau; \bar{\tau}), \tau)$, $\bar{\mu}(r_n(\tau; \bar{\tau}), \tau)$ and their asymptotic values $\lambda_0(\mathcal{R}(\tau - \bar{\tau}))$, $\bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau}))$.

Notation 48 We will write $f(\tau; \bar{\tau}) = O(g(\tau; \bar{\tau}))$ to indicate that $|f(\tau; \bar{\tau})| \leq Cg(\tau; \bar{\tau})$ for $\tau \geq T$, where C is a constant independent of T , τ , $\bar{\tau}$, L , r but perhaps depending on y_0 , R_{\max} .

The function $g(\tau)$ used in the Notation 48 will be usually $\frac{1}{1+(\tau-\bar{\tau})^m}$, $e^{-a\tau}$ for some $m > 0$, $a > 0$.

Lemma 49 Let λ_0 , $\bar{\mu}_0$ be given by (5.36), (5.37). Suppose that $\bar{D} \in \mathcal{Y}_{L,T,a}$ and define λ , $\bar{\mu}$, R_0 as in (5.111), (5.114). Let $r_n(\tau; \bar{\tau})$, $w_n(\tau; \bar{\tau})$ defined by means of (6.2)-(6.4) and ρ_n^* , z_n be given by (6.7). Then:

$$\begin{aligned} & \lambda(\tau, r_n(\tau; \bar{\tau})) - \lambda_0(\mathcal{R}(\tau - \bar{\tau})) \\ &= \frac{1}{2} \frac{\rho_n^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} - \frac{1}{2} \frac{\rho_n^*(\tau; \bar{\tau}) - (R_0(\tau, r) - \frac{2R_{\max}}{3})}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} + R_1(\tau; \bar{\tau}) \end{aligned} \quad (6.8)$$

where:

$$|R_1(\tau; \bar{\tau})| \leq C \left(\left(\frac{\rho_n^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right)^2 + \left(\frac{\rho_n^*(\tau; \bar{\tau}) - (R_0(\tau, r) - \frac{2R_{\max}}{3})}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} \right)^2 \right) \quad (6.9)$$

with C depending on y_0 but independent of L . We have also:

$$\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - \bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau})) = \frac{1}{2} \frac{\rho_n^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} - \frac{1}{2} \frac{\rho_n^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} + R_2(\tau; \bar{\tau}) \quad (6.10)$$

with

$$|R_2(\tau; \bar{\tau})| \leq C \left[\left(\frac{\rho_n^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right)^2 + e^{-\delta\tau} \right]$$

where \mathcal{R} is as in Subsection 5.4 and $a > 0$ is as in Lemma 27 (cf. also (5.64)). The constant C depends on y_0 but is independent of L .

Moreover, the following estimate holds:

$$0 \leq \lambda(\tau, r_n(\tau; \bar{\tau})) \leq \frac{C}{r_n(\tau; \bar{\tau})} \quad , \quad T \leq \tau \leq \bar{\tau} \quad (6.11)$$

Proof. Using Lemmas 27, 44 we obtain:

$$\int_{r_+(\tau)}^r 8\pi^2 \tilde{E}(\tau, \xi) \exp(-\lambda(\tau, \xi)) \bar{D}(\tau, \xi) d\xi \leq CLe^{-2\tau} \quad , \quad \tau \geq T \quad (6.12)$$

where C does not depend on L . Using (5.113), (4.7) in Theorem 22, Proposition 23 and (6.12) we obtain:

$$\left| R_0(\tau, r) - \frac{2R_{\max}}{3} \right| \leq Ce^{-\delta\tau} + CLe^{-2\tau} \quad , \quad \tau \geq T \quad (6.13)$$

where $\delta > 0$ is as in Theorem 22. Therefore the difference $|R_0(\tau, r) - \frac{2R_{\max}}{3}|$ is small if T is large enough. Using (5.111) we obtain:

$$\begin{aligned} & \lambda(\tau, r_n(\tau; \bar{\tau})) - \lambda_0(\mathcal{R}(\tau - \bar{\tau})) \\ &= \frac{1}{2} \log \left(\frac{r_n(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right) + \frac{1}{2} \log \left(\frac{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}}{r_n(\tau; \bar{\tau}) - R_0(\tau, r)} \right) \\ &= \frac{1}{2} \log \left(1 + \frac{\rho^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right) - \frac{1}{2} \log \left(1 + \frac{\rho^*(\tau; \bar{\tau}) - (R_0(\tau, r) - \frac{2R_{\max}}{3})}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} \right) \end{aligned}$$

and (6.8), (6.9) follow using Taylor's Theorem.

We now compute, using (5.112) and (5.37), the difference:

$$\begin{aligned} & \bar{\mu}(\tau, r(\tau; \bar{\tau})) - \bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau})) \\ &= \bar{\mu}(\tau, r_+(\tau)) + \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{4\pi^2 \exp(-\lambda(\tau, \xi)) \bar{w}^2(\tau, \xi) \bar{D}(\tau, \xi)}{\tilde{E}(\tau, \xi)} \frac{d\xi}{\xi - R_0} \\ &+ \frac{1}{2} \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{d\xi}{(\xi - R_0(\tau, \xi))} - \frac{1}{2} \log \left(\frac{r(\tau; \bar{\tau})}{r_+(\tau)} \right) \\ &- \bar{\mu}_0(\infty) - \frac{1}{2} \log \left(1 - \frac{2R_{\max}}{3\mathcal{R}(\tau - \bar{\tau})} \right) \end{aligned}$$

where $\bar{\mu}_0(\infty) = \log \left(\frac{R_{\max} \sqrt{3(1-y_0^2)}}{y_0} \right)$. Then, adding and subtracting in the right-hand side the term $\frac{1}{2} \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{d\xi}{(\xi - \frac{2R_{\max}}{3})}$ we obtain, after some computa-

tions:

$$\begin{aligned}
& \bar{\mu}(\tau, r(\tau; \bar{\tau})) - \bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau})) \\
&= \bar{\mu}(\tau, r_+(\tau)) + \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{4\pi^2 \exp(-\lambda(\tau, \xi)) \bar{w}^2(\tau, \xi) \bar{D}(\tau, \xi)}{\tilde{E}(\tau, \xi)} \frac{d\xi}{\xi - R_0} \\
&+ \frac{1}{2} \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{(R_0(\tau, \xi) - \frac{2R_{\max}}{3}) d\xi}{(\xi - \frac{2R_{\max}}{3} - (R_0(\tau, \xi) - \frac{2R_{\max}}{3})) (\xi - \frac{2R_{\max}}{3})} \\
&+ \frac{1}{2} \log \left(\frac{r(\tau; \bar{\tau}) - \frac{2R_{\max}}{3}}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} \right) + \frac{1}{2} \log \left(\frac{r_+(\tau)}{r_+(\tau) - \frac{2R_{\max}}{3}} \right) \\
&- \frac{1}{2} \log \left(\frac{r(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right) - \bar{\mu}_0(\infty)
\end{aligned}$$

Using (5.3) and (4.6), Proposition 23 that implies $\hat{U}\left(\frac{r_+(t)}{(-t)}\right) = \log\left(\frac{\sqrt{1-y_0^2}}{y_0}\right) + O((-t)^\delta)$ as well as the identity $\log\left(\frac{R_{\max}\sqrt{1-y_0^2}}{y_0}\right) - \bar{\mu}_0(\infty) = -\frac{\log(3)}{2}$ we obtain:

$$\begin{aligned}
& \bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - \bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau})) \tag{6.14} \\
&= \log\left(\frac{r_+(t)}{R_{\max}}\right) + \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{4\pi^2 \exp(-\lambda(\tau, \xi)) \bar{w}^2(\tau, \xi) \bar{D}(\tau, \xi)}{\tilde{E}_1(\tau, \xi)} \frac{d\xi}{\xi - R_0} \\
&+ \frac{1}{2} \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{(R_0(\tau, \xi) - \frac{2R_{\max}}{3}) d\xi}{(\xi - \frac{2R_{\max}}{3} - (R_0(\tau, \xi) - \frac{2R_{\max}}{3})) (\xi - \frac{2R_{\max}}{3})} \\
&+ \frac{1}{2} \log \left(\frac{r_n(\tau; \bar{\tau}) - \frac{2R_{\max}}{3}}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} \right) + \frac{1}{2} \log \left(\frac{r_+(\tau)}{3(r_+(\tau) - \frac{2R_{\max}}{3})} \right) \\
&- \frac{1}{2} \log \left(\frac{r_n(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right) + O(e^{-\delta\tau}) \tag{6.15}
\end{aligned}$$

Using (6.13) we obtain:

$$\begin{aligned}
& \left| \frac{1}{2} \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{(R_0(\tau, \xi) - \frac{2R_{\max}}{3}) d\xi}{(\xi - \frac{2R_{\max}}{3} - (R_0(\tau, \xi) - \frac{2R_{\max}}{3})) (\xi - \frac{2R_{\max}}{3})} \right| \\
&\leq C[e^{-\delta\tau} + Le^{-2\tau}] \tag{6.16}
\end{aligned}$$

On the other hand, using (5.46), (5.56), (5.57), and Lemmas 27, 44 we obtain:

$$\left| \int_{r_+(\tau)}^{r(\tau; \bar{\tau})} \frac{4\pi^2 \exp(-\lambda(\tau, \xi)) \bar{w}^2(\tau, \xi) \bar{D}(\tau, \xi)}{\tilde{E}_1(\tau, \xi)} \frac{d\xi}{\xi - R_0} \right| \leq CLe^{-2\tau} \tag{6.17}$$

Using Proposition 23 and Taylor's Theorem we arrive at:

$$\frac{1}{2} \log \left(\frac{r_+(\tau)}{3(r_+(\tau) - \frac{2R_{\max}}{3})} \frac{r_+(\tau)}{R_{\max}} \right) \leq Ce^{-4\tau} \tag{6.18}$$

Plugging (6.16)-(6.18) into (6.14) and using also (6.7) we obtain:

$$\begin{aligned}\bar{\mu}(\tau, r(\tau; \bar{\tau})) - \bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau})) &= \frac{1}{2} \log \left(1 + \frac{\rho^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau}) - \frac{2R_{\max}}{3}} \right) \\ &\quad - \frac{1}{2} \log \left(1 + \frac{\rho^*(\tau; \bar{\tau})}{\mathcal{R}(\tau - \bar{\tau})} \right) + O(e^{-\delta\tau})\end{aligned}$$

and using Taylor's Theorem we arrive at (6.10).

Using (5.111), (6.13) as well as Taylor's Theorem we obtain (6.11) due to the fact that $r_+(\tau) \rightarrow R_{\max}$ as $\tau \rightarrow \infty$. ■

We now write for further reference the differential equations satisfied by $\rho_n^*(\tau; \bar{\tau})$ and $z_n(\tau; \bar{\tau})$ defined in (6.7).

Lemma 50 *Suppose that $r_n(\tau; \bar{\tau})$ and $w_n(\tau; \bar{\tau})$ solve (6.2)-(6.4). Then, the functions $\rho_n^*(\tau; \bar{\tau})$, $z_n(\tau; \bar{\tau})$ defined in (6.7) solve the following system of equations for any $(\tau; \bar{\tau}) \in \mathcal{U}(T)$:*

$$\begin{aligned}\frac{\partial \rho_n^*(\tau; \bar{\tau})}{\partial \tau} &= \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - 2\lambda(\tau, r_n(\tau; \bar{\tau}))} w_n(\tau; \bar{\tau}) r_n(\tau; \bar{\tau})}{\sqrt{1 + e^{-2\lambda(\tau, r_n(\tau; \bar{\tau}))} (w_n(\tau; \bar{\tau}))^2 (r_n(\tau; \bar{\tau}))^2}} \\ &\quad - \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau)) - 2\lambda_0(\mathcal{R}(\tau))} \mathcal{W}(\tau - \bar{\tau}) \mathcal{R}(\tau - \bar{\tau})}{\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau - \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}} \\ \rho_n^*(\bar{\tau}; \bar{\tau}) &= r_+(\tau) - R_{\max} \\ \frac{\partial z_n(\tau; \bar{\tau})}{\partial \tau} &= - \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau}))} (e^{2\lambda(\tau, r_n(\tau; \bar{\tau}))} - 1)}{2\sqrt{1 + e^{-2\lambda(\tau, r_n(\tau; \bar{\tau}))} (r_n(\tau; \bar{\tau}))^2 (w_n(\tau; \bar{\tau}))^2}} A[r_n(\tau; \bar{\tau}), w_n(\tau; \bar{\tau})] \\ &\quad + \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau))} (e^{2\lambda_0(\mathcal{R}(\tau - \bar{\tau}))} - 1)}{2\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau - \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}} A[\mathcal{R}(\tau - \bar{\tau}), \mathcal{W}(\tau - \bar{\tau})] \\ &\quad + \frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau}))}}{(r_n(\tau; \bar{\tau}))^2 \sqrt{1 + e^{-2\lambda(\tau, r_n(\tau; \bar{\tau}))} (r_n(\tau; \bar{\tau}))^2 (w_n(\tau; \bar{\tau}))^2}} \\ &\quad - \frac{e^{\bar{\mu}_0(\mathcal{R}(\tau - \bar{\tau}))}}{(\mathcal{R}(\tau - \bar{\tau}))^2 \sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau - \bar{\tau}))} (\mathcal{R}(\tau - \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}}\end{aligned}\tag{6.19}$$

$$z_n(\tau; \bar{\tau}) = \frac{e^{\lambda(\tau, r(\tau; \bar{\tau}))}}{(-\bar{t})} V_1 \left(\frac{r_+(\tau)}{(-\bar{t})} \right) + \frac{6y_0 \sqrt{(1 - y_0^2)}}{(1 - 4y_0^2) R_{\max}}$$

where we define:

$$A[r, w] = \left[\frac{1}{(r(\tau; \bar{\tau}))^2} + 2e^{-2\lambda(r(\tau; \bar{\tau}), \tau)} (w(\tau; \bar{\tau}))^2 \right]$$

Proof. The Lemma follows just subtracting (5.40), (5.41), (5.42) from (6.2), (6.3), (6.4) respectively. ■

Remark 51 *These equations are satisfied at any point of $\mathcal{U}(T)$, not only a.e.*

Lemma 52 *Given $L > 0$, there exists $T_0 = T_0(L) > 0$ with the following property. Suppose that $T \geq T_0$. Let $\bar{D} \in \mathcal{Y}_{L,T,a}$. Let (r_n, w_n, D_n) be as in (6.1) (cf. also (6.2)-(6.4), (6.6)). Then, the following estimate holds:*

$$\sup_{\{T \leq \tau \leq \bar{\tau}\}} L \left(\frac{|\rho_n^*(\tau; \bar{\tau})|}{1 + (\bar{\tau} - \tau)} + |z_n(\tau; \bar{\tau})| \right) \leq \frac{1}{5}$$

Proof. We first remark that, as long as we have the inequality

$$w_n(\tau; \bar{\tau}) \leq -\frac{\Gamma(y_0, R_{\max})}{2} \quad (6.20)$$

with $\Gamma(y_0, R_{\max})$ as in Proposition 24, the following estimate holds

$$\left| \frac{\partial w_n(\tau; \bar{\tau})}{\partial \tau} \right| \leq \frac{C}{(r_n(\tau; \bar{\tau}))^2} \quad (6.21)$$

Indeed, due to (6.20) we have $\sqrt{1 + e^{-2\lambda(r_n(\tau; \bar{\tau}), \tau)} (r_n(\tau; \bar{\tau}))^2 (w_n(\tau; \bar{\tau}))^2} \geq C r_n(\tau; \bar{\tau})$. Combining this estimate also with the fact that (6.11) implies $(e^{2\lambda(r_n(\tau; \bar{\tau}))} - 1) \leq \frac{C}{r_n(\tau; \bar{\tau})}$ we obtain (6.21).

Using (6.19) as well as Lemma 49 we can derive a system of equations which depends, to the leading order, linearly on (ρ_n^*, z_n) and contains source terms proportional to $e^{-\delta\tau}$. Then z_n, ρ_n^* can be made arbitrarily small in any bounded region if T is large enough. In particular this implies that on such a time interval time the inequality (6.20) holds. Moreover, due to Proposition 24 we can obtain that for any $R > 0$ there exists \hat{T} such that, for $(\bar{\tau} - \tau) \geq L$ we have $r_n(\tau; \bar{\tau}) \geq R$ if T_0 is sufficiently large. Choosing R large enough, as well as estimate (5.65) it would then follow that the change of $w_n(\tau; \bar{\tau})$ can be made arbitrarily small in the whole set of values $T \leq \tau \leq \bar{\tau}$. A similar estimate can be proved for $\mathcal{W}(\tau)$. The estimate (6.20) would be proved by means of a continuation argument to that set of values. It then follows that

$$|z_n(\tau; \bar{\tau})| \leq \frac{1}{10L} \quad (6.22)$$

if T_0 is sufficiently large.

In order to obtain the estimate of ρ_n^* we use the first equation of (6.19). The differences of functions containing the fields $\lambda, \bar{\mu}$ or their limit values $\lambda_0, \bar{\mu}_0$ are small if T_0 is large. Actually these differences could contain terms like $\left(\frac{\rho_n^*}{\mathcal{R}}\right)^2$ that are smaller than the expected contribution of $\frac{\rho_n^*}{1 + (\bar{\tau} - \tau)}$. Notice that in terms like the ones coming from (6.8), (6.10) we obtain some contributions with the form $C \frac{\rho_n^*}{\mathcal{R}^2}$. The contribution due to these terms can be estimated using

Gronwall arguments, and due to the integrability of $\frac{1}{\mathcal{R}^2}$ the corresponding effect in ρ_n^* would be small. Some terms that must be estimated carefully are the differences of the form:

$$\frac{r_n(\tau; \bar{\tau})}{\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (r_n(\tau; \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}} - \frac{\mathcal{R}(\tau - \bar{\tau})}{\sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau - \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}}$$

The differences or the other terms can be estimated easily. These differences give terms of the form:

$$\frac{r_n(\tau; \bar{\tau}) J_1 - \mathcal{R}(\tau - \bar{\tau}) J_2}{J_1 J_2}$$

where:

$$J_1 = \sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau - \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}$$

$$J_2 = \sqrt{1 + e^{-2\lambda_0(\mathcal{R}(\tau))} (r_n(\tau; \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2}$$

Taking conjugates (of the roots) we obtain differences with orders of magnitude:

$$\frac{\rho_n^*(\tau; \bar{\tau})}{(\mathcal{R}(\tau - \bar{\tau}))^3}$$

Notice that terms like $(r_n(\tau; \bar{\tau}))^2 e^{-2\lambda_0(\mathcal{R}(\tau))} (\mathcal{R}(\tau - \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2$ and $(\mathcal{R}(\tau - \bar{\tau}))^2 e^{-2\lambda_0(\mathcal{R}(\tau))} (r_n(\tau; \bar{\tau}))^2 (\mathcal{W}(\tau - \bar{\tau}))^2$ cancel out. Therefore, these terms do not modify the order of magnitude of $\rho_n^*(\tau; \bar{\tau})$. We then obtain that, taking T_0 large enough, we would obtain the estimate:

$$|\rho_n^*(\tau; \bar{\tau})| \leq \frac{1 + (\bar{\tau} - \tau)}{10L} \quad (6.23)$$

Combining (6.22), (6.23) we conclude the proof of the Lemma. \blacksquare

As a next step we obtain estimates for the derivatives of the functions $r_n(\tau; \bar{\tau})$, $w_n(\tau; \bar{\tau})$.

Lemma 53 *Given $L > 0$, there exists $T_0 = T_0(L) > 0$ such that, for any $\bar{D} \in \mathcal{Y}_{L,T,a}$ if we define (r_n, w_n, D_n) as in Subsection 6.1 the following estimate holds a.e. $(\tau, \bar{\tau}) \in \mathcal{U}(T)$:*

$$L \left[\left| \frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}} + \mathcal{R}'(\tau - \bar{\tau}) \right| + \left| \frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}} + \mathcal{W}'(\tau - \bar{\tau}) \right| \right] \leq \frac{1}{5} \quad (6.24)$$

Proof. To this end we need to differentiate (5.30)-(5.32) with respect to $\bar{\tau}$. The formulas for the derivatives are rather long, but we can estimate the form of the resulting linear equations for large values of $r_n(\tau; \bar{\tau})$. This is the only range of values that we need to estimate in detail, since for bounded values of $(\bar{\tau} - \tau)$ we can obtain estimates using Gronwall-like arguments. It is worth to examine the type of terms that we obtain in the equations (6.2)-(6.4).

First, we have terms coming from the derivatives of $\bar{\mu}$, λ . We differentiate with respect to $\bar{\tau}$. Therefore, we do not need to differentiate with respect to the terms in the previous iteration, since they depend on τ . We need to estimate the derivatives of the functions $\lambda(\tau, r)$, $\bar{\mu}(\tau, r)$ which can be computed by means of:

$$\lambda(\tau, r) = \frac{1}{2} \log \left(\frac{r}{r - R_0(\tau, r)} \right) = \frac{1}{2} \log(r) - \frac{1}{2} \log(r - R_0(\tau, r))$$

$$\begin{aligned} \bar{\mu}(\tau, r) &= \bar{\mu}(\tau, r_+(\tau)) + \int_{r_+(\tau)}^r \frac{4\pi^2 v_1^2(\tau, \xi) B_1(\tau, \xi)}{\tilde{E}_1(\tau, \xi)} \frac{d\xi}{\xi - R_0} \\ &+ \frac{1}{2} \int_{r_+(\tau)}^r \frac{d\xi}{(\xi - R_0(\tau, \xi))} - \frac{1}{2} \log \left(\frac{r}{r_+(\tau)} \right) \end{aligned}$$

It is easily seen that the derivative of both functions with respect to r decreases like $\frac{1}{r^2}$. Therefore, this yields in the equations for $r_n(\tau; \bar{\tau})$ and $w_n(\tau; \bar{\tau})$ terms decreasing like $\frac{1}{1+(\bar{\tau}-\tau)^2} \frac{\partial r}{\partial \bar{\tau}}$. On the other hand, the derivatives of $r_n(\tau; \bar{\tau})$ in both equations result in terms decreasing rather fast too. The reason is that in the first term the asymptotics as $r \rightarrow \infty$ is like a constant. The correction is like $\frac{1}{r}$, and the derivative gives again terms of order $\frac{1}{r^2}$. In the first equation this results in terms like $\frac{1}{1+(\bar{\tau}-\tau)^2}$ multiplied by $\frac{\partial r}{\partial \bar{\tau}}$ and in the equation for w this results in terms like $\frac{1}{1+(\bar{\tau}-\tau)^3}$ multiplied by $\frac{\partial r}{\partial \bar{\tau}}$.

We now consider the derivatives of w . They give terms that are multiplied at least by $\frac{1}{1+(\bar{\tau}-\tau)^2}$ in the second equation, since we have basically the same contributions as in the equation without derivatives. On the other hand, the effect of the terms containing w in the first equation is more subtle. Notice that for large values of r the function

$$\frac{e^{\bar{\mu}(\tau, r_n(\tau; \bar{\tau})) - 2\lambda(\tau, r_n(\tau; \bar{\tau}))} w_n(\tau; \bar{\tau}) r_n(\tau; \bar{\tau})}{\sqrt{1 + e^{-2\lambda(\tau, r_n(\tau; \bar{\tau}))} (w_n(\tau; \bar{\tau}))^2 (r_n(\tau; \bar{\tau}))^2}}$$

is independent of w . This means that this dependence does not appear for large values. Using Taylor we obtain a dependence with the form $\frac{G(w)}{(r_n(\tau; \bar{\tau}))^2}$ at least.

This results in terms with the form $\frac{1}{1+(\bar{\tau}-\tau)^2} \frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$. Therefore, the linearized equation has the following form:

$$\begin{aligned} \frac{\partial}{\partial \tau} \left(\frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}} \right) &= O \left(\frac{1}{1+(\bar{\tau}-\tau)^2} \right) \frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}} + O \left(\frac{1}{1+(\bar{\tau}-\tau)^2} \right) \frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}} \\ \frac{\partial}{\partial \tau} \left(\frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}} \right) &= O \left(\frac{1}{1+(\bar{\tau}-\tau)^3} \right) \frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}} + O \left(\frac{1}{1+(\bar{\tau}-\tau)^2} \right) \frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}} \end{aligned}$$

where these formulas must be understood in weak form, as in the proof of Lemma 31. Therefore the functions $\frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$, $\frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$ are close to constant. They are

determined then by the boundary values. We use the equations:

$$r_n(\tau; \bar{\tau}) = r_+(\tau) \quad , \quad w_n(\tau; \bar{\tau}) = \frac{e^{\lambda(\tau, r_n(\tau; \bar{\tau}))}}{(-\bar{t})} V_1 \left(\frac{r_+(\tau)}{(-\bar{t})} \right)$$

Then:

$$\begin{aligned} \frac{\partial r_n(\tau; \bar{\tau})}{\partial \tau} + \frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}} &= O(e^{-\delta \bar{\tau}}) \\ \frac{\partial w_n(\tau; \bar{\tau})}{\partial \tau} + \frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}} &= O(e^{-\delta \bar{\tau}}) \end{aligned}$$

We could have much better estimates using the estimates for the self-similar solution. On the other hand we can compute $\frac{\partial r_n(\tau; \bar{\tau})}{\partial \tau}$, $\frac{\partial w_n(\tau; \bar{\tau})}{\partial \tau}$ using the differential equation itself. This would give an approximation of order:

$$\begin{aligned} \frac{\partial r_n(\tau; \bar{\tau})}{\partial \tau} &= \mathcal{R}'(\bar{\tau} - \tau) + O(e^{-\delta \bar{\tau}}) \\ \frac{\partial w_n(\tau; \bar{\tau})}{\partial \tau} &= \mathcal{W}'(\bar{\tau} - \tau) + O(e^{-\delta \bar{\tau}}) \end{aligned}$$

We then obtain, using standard continuous dependence results, that $\frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$ and $\frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$ can be approximated for $(\bar{\tau} - \tau)$ of order one, by means of $-\mathcal{R}'(\tau - \bar{\tau})$ and $-\mathcal{W}'(\tau - \bar{\tau})$ respectively. For large values of $(\bar{\tau} - \tau)$ the integrable decay of the terms in the linearized equations imply a small change of the values of $\frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$ and $\frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}}$. We then have that:

$$\left| \frac{\partial r_n(\tau; \bar{\tau})}{\partial \bar{\tau}} + \mathcal{R}'(\tau - \bar{\tau}) \right| \quad , \quad \left| \frac{\partial w_n(\tau; \bar{\tau})}{\partial \bar{\tau}} + \mathcal{W}'(\tau - \bar{\tau}) \right|$$

are uniformly small if T is large enough. Therefore, the corresponding terms in the norm can be estimated by a quantity arbitrarily small if T is large.

It is important to take into account that we can differentiate the equations satisfied by $r_n(\tau; \bar{\tau})$, $w_n(\tau; \bar{\tau})$ only *a.e.* The argument must be done in the integrated version of the differential equations satisfied by $r_n(\tau; \bar{\tau})$, $w_n(\tau; \bar{\tau})$. We take derivatives with respect to $\bar{\tau}$ which can be introduced inside the integrals. The estimates are done then using a Gronwall argument and this gives estimates *a.e.* ■

Lemma 54 *Given $L > 0$, there exists $T_0 = T_0(L) > 0$ such that, for any $\bar{D} \in \mathcal{Y}_{L,T,a}$ if we define (r_n, w_n, D_n) as in Subsection 6.1 we have:*

$$\frac{\partial r_n}{\partial \tau}(\tau; \bar{\tau}) \leq -\frac{1}{L} \quad , \quad a.e. \quad (\tau, \bar{\tau}) \in \mathcal{U}(T)$$

Proof. It is just a consequence of (6.2) and Lemma 52. ■

Finally we estimate the function D_n using (6.6) as well as the estimates obtained for the functions r_n , w_n .

Lemma 55 *There exists $L_0 > 0$, such that, for any $L > L_0$, there exists $T_0 = T_0(L)$ such that, for $T > T_0$, given $\bar{D} \in \mathcal{Y}_{L,T,a}$ if we define (r_n, w_n, D_n) as in (6.1) (cf. also (6.2)-(6.4), (6.6)) we have:*

$$D_n(\tau; \bar{\tau}) \exp(2\bar{\tau}) \leq 1 \quad , \quad a.e. \quad (\tau, \bar{\tau}) \in \mathcal{U}(T)$$

Proof. We need to obtain estimates for D_n . To this end we use (6.6) which is satisfied *a.e.* in $\mathcal{U}(T)$. It is relevant to examine the dominant terms in this equation as $r \rightarrow \infty$. The term $\partial_r \bar{w}_n(r(\tau; \bar{\tau}), \tau) = (\frac{\partial w_n}{\partial \bar{\tau}}(\tau, \bar{\tau})) / (\frac{\partial r_n}{\partial \bar{\tau}}(\tau, \bar{\tau}))$ is bounded due to (6.24). On the other hand, using the field equations (5.106), (5.107) we can estimate the terms $\bar{\mu}_r(r_n(\tau; \bar{\tau}), \tau)$, $\lambda_r(r_n(\tau; \bar{\tau}), \tau)$ in (6.6). The contribution due to terms in (5.106), (5.107) containing \bar{D} can be estimated, using (5.64), as $Ce^{-2\tau}e^{-ar(\tau; \bar{\tau})}$. On the other hand, the equations (5.106), (5.107) also yield terms $\frac{(1-e^{2\lambda})}{r}$ which can be estimated as $\frac{C}{\bar{\tau}^2}$ using (6.11). Combining all these estimates for $\partial_r \bar{w}_n(r(\tau; \bar{\tau}), \tau)$, $\bar{\mu}_r(r_n(\tau; \bar{\tau}), \tau)$, $\lambda_r(r_n(\tau; \bar{\tau}), \tau)$ and using also Proposition 24 and Lemma 52 we can then rewrite (6.6) as:

$$\frac{\partial D_n(\tau; \bar{\tau})}{\partial \tau} = K(\tau; \bar{\tau}) D_n(\tau; \bar{\tau}) \quad , \quad |K(\tau; \bar{\tau})| \leq \frac{C}{1 + (\bar{\tau} - \tau)^2} \quad (6.25)$$

for some constant C independent of L, T . Using now the boundary condition $D_n(\bar{\tau}; \bar{\tau}) = (-\bar{t}) b_1 \left(\frac{r_+(\bar{\tau})}{(-\bar{t})} \right) \exp(\lambda(r_n(\bar{\tau}; \bar{\tau}), \tau))$ (cf. (6.6)) and using (4.10) as well as the fact that $\gamma(y_0) > 2$ if y_0 is sufficiently small (cf. Remark 26), we obtain, integrating (6.25):

$$0 \leq D_n(\tau; \bar{\tau}) \leq e^{-2\bar{\tau}} \quad , \quad a.e. \quad (\tau, \bar{\tau}) \in \mathcal{U}(T) \quad (6.26)$$

if T is sufficiently large. ■

Proof of Proposition 46. Due to Lemma 27 and (5.70) if we prove that the function (r_n, w_n, D_n) is in the space $\mathcal{X}_{L,T}$ if L and $T_0(L)$ are sufficiently large, we would obtain that the corresponding function $\bar{D}_n \in \mathcal{Y}_{L,T,a}$ if we assume that L is large enough. We will then check that $(r_n, w_n, D_n) \in \mathcal{X}_{L,T}$. The fact that D_n satisfies (5.55) follows from Lemma 55. The inequalities in (5.56) are satisfied by the corresponding functions ρ_n^* , z_n associated to r_n, w_n due to Lemma 52. The inequalities (5.58) are a consequence of Lemma 53. The estimate (5.59) follows from Lemma 54. The inequalities (5.60) follow from (6.2), (6.3), (6.4), (6.6) if L is chosen sufficiently large (independently on T), since the right hand side of (6.2), (6.3), (6.4), (6.6) is uniformly bounded by a constant independent of T if T is large.

The fact that r_n satisfies (5.61) follows by construction (cf. (6.2)) as well as the uniform derivative estimates for r_n . ■

6.3 Weak continuity and compactness of the operator \mathcal{T} .

We need to prove that the operator \mathcal{T} is continuous and compact in the topology of the space $\mathcal{Y}_{L,T,a}$.

Lemma 56 *The operator \mathcal{T} defined in (6.1) is continuous and compact in the topology of the space $\mathcal{Y}_{L,T,a}$ defined in Subsection 5.9.*

Proof. This result is a consequence of the identity (5.75). The functions r_n , w_n depend continuously on \bar{D} in the topology uniform in t and weak in r defined in Section 5.9. Suppose that $\mathcal{T} : \bar{D} \rightarrow \bar{D}_n$. Then:

$$\bar{D}_n(t, r_n(t, \bar{t})) \frac{\partial r_n(t, \bar{t})}{\partial \bar{t}} = D_n(t, \bar{t}) \frac{\partial r_n(t, \bar{t})}{\partial \bar{t}} = D_n(\bar{t}, \bar{t}) \frac{\partial r_n(\bar{t}, \bar{t})}{\partial \bar{t}} = \omega(\bar{t}) \quad (6.27)$$

In order to study the continuity of the operator \mathcal{T} we need to study its action with respect to test functions. Notice that:

$$\begin{aligned} \int_{r_+(t)}^{\infty} \bar{D}_n(t, r) \varphi(t, r) dr &= \int_t^{\infty} \bar{D}_n(t, r_n(t, \bar{t})) \frac{\partial r_n(t, \bar{t})}{\partial \bar{t}} \varphi(t, r_n(t, \bar{t})) d\bar{t} \quad (6.28) \\ &= \int_t^{\infty} \omega(\bar{t}) \varphi(t, r_n(t, \bar{t})) d\bar{t} \end{aligned}$$

The continuity of the functions r_n in the continuous-weak topology in \bar{D} implies the continuity of \mathcal{T} .

On the other hand, in order to prove compactness of the operator \mathcal{T} we need to prove equicontinuity of $\int_{r_+(t)}^{\infty} \bar{D}_n(t, r) \varphi(t, r) dr$. To this end we use the identity (6.28). The function $\gamma(\bar{t})$ is smooth. The equicontinuity of the functional then follows from the equicontinuity of $r_n(t, \bar{t})$ with respect to t , which is a consequence of the differential equation satisfied by this function. The right-hand side is uniformly bounded and therefore we have equicontinuity. This gives the desired compactness. ■

6.4 Fixed point argument. End of the proof of Theorem 19.

End of the proof of Proposition 39. This result follows from Schauder's fixed point Theorem (cf. [10]) combined with Lemma 56 as well as the fact that every fixed point of the operator \mathcal{T} allows us to obtain a solution of (5.21)-(5.24) in the sense of characteristics by means of the solution of the characteristic equations (6.2)-(6.6). ■

Proof of Theorems 19 and 20. Theorem 19 follows from Proposition 32 and Proposition 39. Theorem 20 then follows from Proposition 18. ■

7 Geometrical properties of the solution.

In this section we summarize several geometrical properties of the spacetime constructed in the previous sections. In particular we will prove that the corresponding metric is geodesically incomplete. Moreover, we will rewrite the metric in double-null coordinates. This will allow us to clarify the causal relations between the different regions of the spacetime. A consequence of this will

be a proof of the fact that the spacetime obtained does not contain any horizon separating the regions where the curvature of the spacetime is unbounded and the regions at infinity where $r = \infty$. We summarize the results in the following theorem. In this section we follow the common use of the letters (u, v) to denote the double null coordinates. Therefore, v is not the coordinate introduced in (2.12).

Theorem 57 *There exists a diffeomorphism $(t, r) \rightarrow (u, v)$ which transforms the portion of spacetime $\{(t, r, \theta, \varphi) : 0 \leq r < \infty, t_0 < t < 0, \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$ into the region $\{(u, v, \theta, \varphi) : U_-(v) \leq u < 1 \text{ if } v \in (-1, 0], U_+(v) \leq u < 1 \text{ if } v \in [0, 1], \theta \in [0, \pi], \varphi \in [0, 2\pi]\}$ for suitable functions $U_-, U_+ \in C^1([0, 1])$ satisfying*

$$U_-(0) = U_+(0) < 1, \quad U_-(-1) = 1, \quad U_+(1) = 1 \quad (7.1)$$

$$U_- \text{ is a decreasing function, } U_+ \text{ is an increasing function.} \quad (7.2)$$

The metric ds^2 of the spacetime in the coordinates (u, v, θ, φ) has the form:

$$ds^2 = -(\Omega(u, v))^2 dudv + [r(u, v)]^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (7.3)$$

for a suitable function $\Omega(u, v)$. The system of coordinates (u, v) will be denoted as double-null coordinates.

The curves $\{u = \text{const}\}$ and $\{v = \text{const}\}$ are radial light rays. The center $r = 0$ is, in the coordinates (u, v) the line $u = U_+(v)$, $v \in (0, 1)$. On the other hand, the limit $r \rightarrow \infty$, where the spacetime obtained is asymptotically flat, is represented by the line $u = 1$, $-1 = U_-(1) \leq v < U_+(1)$. The point $(u, v) = (U_+(1), 1)$ is a singularity for the spacetime obtained, since the curvature of the metric becomes unbounded.

The spacetime obtained is geodesically incomplete. More specifically, the curve $\{r = 0, t_0 \leq t < 0\}$ is a geodesic along which the proper time is the coordinate t . Therefore the singular point is reached in a finite proper time along this line.

There exists a light ray connecting any point in the spacetime region

$$\{(u, v) : u \in (0, 1), U_-(u) \leq v \leq U_+(u)\}$$

with the set $\{r = \infty\} = \{u = 1, U_-(1) \leq v < U_+(1)\}$. Therefore, no horizon appears in any part of the spacetime considered.

We will use the following asymptotic description of the fields λ and μ .

Lemma 58 *Suppose that ζ is a solution (2.5)-(2.8), (2.16), (2.17) in the sense of Definition 13 as obtained in Theorem 19 and let $\lambda, \bar{\mu}$ the corresponding fields. Then, the following asymptotics holds:*

$$\left| \lambda(\tau, r) - \frac{1}{2} \log \left(\frac{r}{r - R_0(\tau, \infty)} \right) \right| \leq \bar{C} \exp(-ar) \exp(-b\tau) \quad (7.4)$$

$$\left| \bar{\mu}(\tau, r) - \bar{\mu}(\tau, \infty) - \frac{1}{2} \log \left(1 - \frac{2R_0(\tau, \infty)}{3r} \right) \right| \leq \bar{C} \exp(-ar) \exp(-b\tau) \quad (7.5)$$

uniformly in $r \geq R_{\max}$, $\tau \geq \tau_0$ where:

$$\left| R_0(\tau, \infty) - \frac{2R_{\max}}{3} \right| \leq \bar{C} \exp(-b\tau) \quad (7.6)$$

$$|\bar{\mu}(\tau, \infty) - \bar{\mu}_0(\infty)| \leq \bar{C} \exp(-b\tau) \quad , \quad \bar{\mu}_0(\infty) = \log \left(\frac{R_{\max} \sqrt{3(1-y_0^2)}}{y_0} \right) \quad (7.7)$$

Proof. These formulas are a consequence of the fact that \bar{D} satisfies (5.70). We have similar estimates for ρ , p due to (5.13), (5.20). The estimates in the Lemma then follow from Lemma 2. ■

Proof of Theorem 57. In order to construct the double-null coordinates we need to solve the differential equations:

$$0 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2$$

or equivalently:

$$\frac{dr}{dt} = e^{\mu-\lambda} \quad (7.8)$$

$$\frac{dr}{dt} = -e^{\mu-\lambda} \quad (7.9)$$

We now use the fact that the functions λ , μ have the following self-similar form for $r \leq R_+(t)$:

$$\begin{aligned} \mu(t, r) = \lambda(t, r) = 0 \quad \text{if } r \leq y_0(-t) \quad , \quad t_0 \leq t < 0 \\ \mu(t, r) = U(y) \quad , \quad \lambda(t, r) = \Lambda(y) \quad , \quad y = \frac{r}{(-t)} \quad , \quad r \leq R_+(t) \end{aligned} \quad (7.10)$$

On the other hand, λ , μ are given for $r > R_+(t)$ by means of Lemma 44, with (r, w, D) given by the fixed point obtained in Proposition 39. Using (5.2) we can reformulate (7.8) as:

$$\frac{dr}{d\tau} = e^{\bar{\mu}(\tau, r) - \lambda(\tau, r)} \quad (7.11)$$

Using then the asymptotics (7.4)-(7.7), as well as standard ODE arguments, we obtain that for any $v \in [0, 1]$ there exists a unique solution of (7.11) denoted as $r_+(\tau; v)$ and satisfying:

$$r_+(\tau_0 + \tanh^{-1}(v); v) = 0 \quad , \quad v \in [0, 1] \quad (7.12)$$

and for any $v \in (-1, 0]$ there exists a unique solution of (7.11) such that:

$$r_+(\tau_0; v) = -\tanh^{-1}(v) \quad , \quad v \in (-1, 0] \quad (7.13)$$

The function $\tau \rightarrow r_+(\tau; v)$ is increasing. If $r_+(\tau_0; v) > r_+(\tau_0)$ (cf. (5.5), (5.6)) we have that the intersection between the curves $\{r = r_+(\tau; v), \tau\}$,

$\{r = r_+(\tau_0)\}$ is empty. On the other hand, if $r_+(\tau_0; v) \leq r_+(\tau_0)$ it follows from Lemma 58, (5.7) and (7.11) that for $|t_0|$ sufficiently small there is a unique intersection between the curves $\{r = r_+(\tau; v)\}$, $\{r = r_+(\tau_0)\}$. In both cases the function $r_+(\tau; v)$ is globally defined for $\tau \geq \min\{\tau_0 + \tanh^{-1}(v), \tau_0\}$ and $\lim_{\tau \rightarrow \infty} r_+(\tau; v) = \infty$. The family of disjoint curves

$$\{r = r_+(\tau; v) : \tau \geq \tau_0 + \min\{\tau_0 + \tanh^{-1}(v), \tau_0\}\} \text{ with } v \in (-1, -1)$$

covers the whole set $\{\tau \geq \tau_0, r \geq 0\}$. We can then use these curves to define a function $v : \{\tau \geq \tau_0, r \geq 0\} \rightarrow (-1, -1)$ which assigns to each pair (r, τ) a value $v(r, \tau)$.

On the other hand we define functions $r_-(\tau; u)$ by means of the problem:

$$\frac{dr}{d\tau} = -e^{\bar{\mu}(\tau, r) - \lambda(\tau, r)} \quad , \quad \tau \geq \tau_0 \quad , \quad r_-(\tau_0; u) = \tanh^{-1}(u) \quad , \quad u \in [0, 1] \quad (7.14)$$

The function $\tau \rightarrow r_-(\tau; u)$ is decreasing for each $u \in [0, 1]$. Using Lemma 58, (5.7) and (7.14) it follows that there is at most one intersection between the curves $\{r = r_-(\tau; u)\}$, $\{r = r_+(\tau)\}$. Moreover, there exists exactly one intersection for each $u \in [0, 1]$ such that $\tanh^{-1}(u) \geq r_+(\tau_0)$. We then define $\tau_-^*(u) \geq \tau_0$ for $u \geq \tanh(r_+(\tau_0))$ by means of:

$$r_-(\tau_-^*(u); u) = r_+(\tau_-^*(u))$$

In order to obtain $r_-(\tau; u)$ for $\tau > \tau_-^*(u)$ we use the self-similar form $r_-(\tau; u) = e^{-\tau} y_-(\tau; u)$ which yields the following equation for $y_-(\tau; u)$:

$$-y_- + \frac{dy_-}{d\tau} = -e^{U(y_-) - \Lambda(y_-)} \quad , \quad \tau \geq \tau_-^*(u) \quad , \quad y_-(\tau_-^*(u); u) = e^{\tau_-^*(u)} r_+(\tau_-^*(u))$$

The solution of this equation is given by:

$$F_-(e^{\tau_-^*(u)} r_+(\tau_-^*(u))) - F_-(y_-(\tau; u)) = \tau - \tau_-^*(u) \quad (7.15)$$

where:

$$F_-(y) = \int_0^y \frac{d\xi}{e^{U(\xi) - \Lambda(\xi)} - \xi} \quad (7.16)$$

We now remark that the analysis of the self-similar solutions in [17] imply that $F_-(y)$ is well defined and it is increasing if $y > 0$. In particular, it then follows that (7.15) that for any $u \geq \tanh(r_+(\tau_0))$ there exists $\tau_-^{**}(u) > \tau_-^*(u)$, $\tau_-^{**}(u) < \infty$ such that $y_-(\tau_-^{**}(u); u) = 0$. It then follows that the light rays $\{r = r_-(\tau; u)\}$ reach the center $r = 0$ for a finite value of $\tau = \tau_-^{**}(u)$. In particular the disjoint curves $\{r = r_-(\tau; u)\}$, $u \in [0, 1]$ cover the whole domain $\{\tau \geq \tau_0, r \geq 0\}$ and they can be used to define a function $(r, \tau) \rightarrow v(r, \tau)$. We define the functions $U_-(v)$, $U_+(v)$ by means of the identities:

$$r_+(\tau; v) = r_-(\tau; U_+(v)) = 0 \quad , \quad v \in [0, 1] \quad (7.17)$$

$$r_+(\tau_0; v) = r_-(\tau_0; U_-(v)) = -\tanh^{-1}(v) \quad , \quad v \in (-1, 0] \quad (7.18)$$

The properties (7.1), (7.2) then follow from these formulas. Indeed, the key property which need to be checked is:

$$\inf \{u(\tau, r) : v(\tau, r) \geq 1 - \varepsilon\} \rightarrow 1 \quad (7.19)$$

as $\varepsilon \rightarrow 0^+$. This can be seen as follows. We first claim that $\tau_-^{**}(u) \rightarrow \infty$ as $u \rightarrow 1^-$. To see this we just solve the ODE (7.9) for $\tau_0 \leq \tau \leq T$ with the initial condition $r(T) = 0$ and denote the corresponding solution as $\bar{r}(\cdot; T)$. Due to Lemma 58 this solution is defined in the whole interval $\tau \in [\tau_0, T]$. Moreover, we have

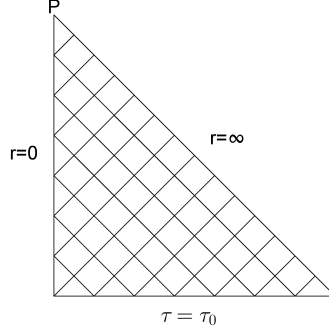
$$\bar{r}(\tau_0; T) \rightarrow \infty \quad (7.20)$$

Indeed, notice that the uniqueness theorem for ODEs imply that the function $T \rightarrow \bar{r}(\tau_0; T)$ is increasing. Suppose that $\lim_{T \rightarrow \infty} \bar{r}(\tau_0; T) = R_\infty < \infty$. Then, due to the uniqueness theorem for ODEs we would obtain that $\tau_-^{**}(u) = \infty$ if $u > \tanh(R_\infty)$, but this contradicts the fact that $r_-(\tau_-^{**}(u); u) < \infty$ for any $u < 1$ as indicated above. Therefore $R_\infty = \infty$ and this implies (7.20). We can now prove (7.19). Notice that the function $f(\varepsilon) = \inf \{u(\tau, r) : v(\tau, r) \geq 1 - \varepsilon\}$ is decreasing in ε and by construction $f(\varepsilon) \leq 1$. Suppose that $\lim_{\varepsilon \rightarrow 0^+} f(\varepsilon) < 1$. Then there exists $\delta > 0$ such that for any $\varepsilon > 0$ there exists at least one point $(\tau_\varepsilon, r_\varepsilon)$ such that $v(\tau_\varepsilon, r_\varepsilon) \geq 1 - \varepsilon$ and $u(\tau_\varepsilon, r_\varepsilon) \leq 1 - \delta$. Moreover, the definition of v implies that $\tau_\varepsilon \geq \tanh^{-1}(v(\tau_\varepsilon, r_\varepsilon)) \geq \tanh^{-1}(1 - \varepsilon)$. However, this is not possible, because due to the monotonicity of the function $u \rightarrow r_-(\tau; u)$ we would have that $\tau_\varepsilon \leq \tau_-^{**}(u(\tau_\varepsilon, r_\varepsilon)) \leq \tau_-^{**}(1 - \delta) < \infty$ and this gives a contradiction if ε is sufficiently small.

Using (7.19) we obtain the identity $U_+(1) = 1$ in (7.1). The rest of the identities in (7.1), (7.2) follow easily from the definitions of U_+, U_- in (7.17), (7.18). ■

Notice that the causal structure of the spacetime obtained shows that no horizon is formed, since any point in the spacetime can send a light ray reaching infinity. However, the singularity obtained is not the type of singularity usually termed a naked singularity. Indeed, we remark that there is not any light ray starting at the singularity and reaching infinity. The spacetime obtained is geodesically incomplete, since the particles at $r = 0$, reach the singularity in a finite proper time. On the other hand, light rays emitted at the center $r = 0$, at times in which the curvatures are arbitrarily large reach infinity, but no light ray emitted from the singular point reaches infinity. The solutions obtained have also a causal structure different from the one associated to the so-called "collapsed cones" which were obtained in [6] for the Einstein equations coupled to a scalar field. The most distinctive feature of the solutions constructed in this paper is the fact that the singularity cannot be reached by any light ray having as starting point any point of the space-time obtained. It turns out, however, that the singular point can be reached by some specific time-like trajectories. In the case of the singularities in [6] it is possible to connect points of the space-time and points of the singularity by means of light-rays. Figure 1 contains a Penrose diagram of the space-time in order to explain the causal structure of

Figure 1: Penrose diagram of the spacetime obtained.



the metric. Notice that the only singular point in this space time is the point P . On the other hand, the point P cannot be reached by any radial light ray, although it can be reached by some time-like curves in finite proper time, for instance $\{r = 0\}$.

Notice that the functions ρ , p become unbounded at the surface $\{r = y_0(-t)\}$, and they are discontinuous at $\{r = R_+(t)\}$. Therefore, the derivatives of the fields λ, μ are unbounded in the surface $\{r = y_0(-t)\}$ and they are discontinuous at $\{r = R_+(t)\}$ (cf. (2.5), (2.6)). In particular the so-called Kretschmann scalar (cf. ([16])) becomes unbounded in a neighbourhood of the set of turning points $\{r = y_0(-t)\}$. Nevertheless, the integrability of the functions p , ρ in a neighbourhood of $\{r = y_0(-t)\}$ suggests that the singularity at the turning points is not a true singularity induced by the nonlinear character of Einstein equations, but that it is more a fictitious type of singularity due to the singular type of matter used (dust-like solutions). A discussion about singularities which are due to the presence of singular behaviours in the matter model under consideration can be found in [12], Subsection 8.4. It is indicated in [12] that the minimal condition that must be requested to a spacetime to consider it singularity free is timelike and null geodesic completeness. From this point of view, it might be considered that the spacetime constructed in this paper is singularity free for $t < 0$. Indeed, a careful analysis of the geodesic equations $\frac{d^2 x_\alpha}{d\zeta^2} + \Gamma_{\alpha\gamma}^\beta \frac{dx^\alpha}{d\zeta} \frac{dx^\gamma}{d\zeta} = 0$ shows that their evolution is well defined for every time-like and null characteristic in a neighbourhood of the surface $\{r = y_0(-t), t < 0\}$. Away from this surface the geodesic completeness follows from the smoothness of the metric.

It is natural to ask in which sense the solution described in this paper represents a singularity of the spacetime which has worse properties than the spacetime for $t = t_0 < 0$, which is already singular, due to the divergence of ρ and p at $r = y_0(-t)$. Seemingly the answer to this question is that the sin-

gular character of the spacetime is only apparent due to collapse of the whole structure towards $r = 0$ as $t \rightarrow 0^-$. More precisely, we can construct several quantities which exhibit this collapsing behaviour as $t \rightarrow 0^-$. One possibility is the following. Suppose that we denote as Kr the Kretschmann scalar given by $\text{Kr} = R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}$ (cf. [16]). It has been seen in [17] that $\text{Kr} \geq 0$. Suppose that we take $A > 0$ sufficiently large, but fixed. It then follows that:

$$\frac{1}{(-t)} \int_0^{A(-t)} (\text{Kr})^\theta dr \geq \frac{C_\theta}{(-t)^{2\theta}} \quad (7.21)$$

for any $\theta \in (0, 1)$ with $C_\theta > 0$. This inequality indicates that the curvature near the self-similar region is divergent in some suitable average sense. Notice that we cannot take $\theta \geq 1$ due to the singular behaviour of ρ, p near $r = y_0(-t)$, since in that case the left-hand side of (7.21) would be infinity. Another quantity which shows the singular character of the spacetime constructed in this paper can be constructed in terms of the so-called Hawking mass m_H . This quantity, and more precisely $\frac{m_H}{r}$ measures the degree of deformation of the spacetime up to some radius r . In the setting of this paper we have (cf. [16]):

$$\frac{2m_H}{r} = 1 - e^{-2\lambda}$$

and given that for small r and t we have $\lambda = \Lambda\left(\frac{r}{(-t)}\right)$, with $\Lambda(0) = \Lambda(\infty) = 0$, $\Lambda(y) = 0$ for $0 \leq y \leq y_0$, $\Lambda(y) > 0$ if $y_0 < y < \infty$, we obtain:

$$\lim_{t \rightarrow 0} \sup_{0 \leq r \leq A(-t)} \frac{m_H}{r} \geq c_1 > 0$$

where $A > y_0$ is a fixed constant. This formula indicates also the presence of a concentration of mass-energy for small r and t . Note that the Hawking mass is continuous at the turning point.

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